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# KEY

TO THE

# INTRODUCTION TO ALGEBRA;

CONTAINING

## SOLUTIONS IN FULL

TO ALL THE EXAMPLES,

WITH

IMPROVED METHODS OF OPERATION IN SEVERAL
PARTS OF THE SUBJECT.

BY J. R. YOUNG,

EMERITUS PROFESSOR OF MATHEMATICS, BELFAST COLLEGE.

BIBLIOTHECA · BODLEIANA

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1854.

181. c. 127.

# PREFACE.

I HAVE endeavoured to make this Key something more than a dry collection of processes, exemplifying the mere mechanism of practical Algebra.

Its main object is, of course, to furnish, with sufficient detail, full-length solutions to all the unworked examples in the book with which it is connected. But I have not failed to take advantage of the opportunity which the preparation of a work of this kind affords, to edge in here and there whatever collateral information I could, consistently with the avowed character of the performance. These supplementary items of knowledge, introduced as occasion offered, merely fill up, as it were, the little interstices that would otherwise remain vacant, and help to render the book in some measure a Companion, as well as a Key, to the former volume. In this way I have been enabled to introduce illustrative remarks and practical improvements, that may be acceptable even to those who may have but little occasion for a KEY, in the sense in which a book so called is usually understood. For instance, I have given some particulars, which I hope may prove interesting to the inquiring student, in reference to Detached Coefficients, to the easier management of Irrational Polynomials, and to the general theory and interpretation of Congeneric Surd Equations — topics but very briefly touched upon in the ALGEBRA itself, but which the learner will find to be deserving of more special notice. The last-mentioned subject in particular—the subject of Congeneric Surd Equations—is, I think, here put in a much more clear and satisfactory light than it has hitherto appeared in; and it will be found, I hope, to be now as little chargeable with obscurity and uncertainty as the other parts of Algebra.

I may mention, too, that in reference to the important topic of *Numerical Equations*, I have largely illustrated a very efficient practical method of finding the leading figure of a root—a method announced by me a few years ago, but as yet but little known. I am not an advocate for the introduction into elementary books of this kind, of every thing which authors may consider to be original. Such books are scarcely the fitting places for innovations in science, however original, and, indeed, however valuable they may be, unless they very palpably abridge labour, or smooth down difficulties.

But to find the first figure of a real root of a numerical equation, is a preliminary indispensable to every method of actual development. Hitherto little or no aid has been supplied, by books on Algebra, to guide the learner in his search for it; I was therefore under the necessity of giving him directions of my own, or else leaving him without any resource but that of chance-trial or guess-work. Of course, I am speaking here of elementary school-books, and not of works devoted expressly and exclusively to the general theory and solution of Equations.

As I am anxious that the method here adverted to should be placed before the student with sufficient fulness to enable him to gain a comprehensive view of its advantages, I have introduced some supplementary instructions respecting it in a NOTE at the end. I have also given, in the form of Notes, some additional information on two other topics: on the general theory of Elimination in Equations of the first Degree, and on the Exponential Theorem. This latter subject is treated in the note referred to in a peculiar way; but the student will perceive that brevity of investigation and completeness of result are both combined in the method adopted; and from the satisfactory specimen thus furnished, he may be led to invest the apparently insignificant symbol 0 with more importance, among the implements of analytical research, than is usually accorded to it—though not more than is justly its due. As the opportunity offered, and as a page or two of space was still at my disposal, I have followed up this investigation with some particulars of use only in more advanced inquiries, though I have not failed to make them the occasion for mentioning a principle—connected with the Multiplication of Series—which will be found to shorten the work of certain questions often met with in common Algebra, namely, the finding a specified term in the product of two given algebraical polynomials: and with an illustration of this principle the volume ends.

I have only further to add, that all the examples and questions in the Algebra I have worked out myself—none have been copied; and that I entertain the hope that the new edition of the Algebra, now carefully revised and thoroughly corrected, if studied in connexion with this volume of Solutions, will be found sufficient to enable an otherwise unaided student to acquire a sound and accurate acquaintance with the scientific principles of Algebra, both in theory and in practice.

J. R. YOUNG.

London, 1854.

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# KEY

TO

# THE INTRODUCTION TO ALGEBRA.

---

THIS Key is written with a twofold purpose: one object—and the principal object—is to supply to the self-taught student the absence of an instructor; the other is to furnish occasional aid to teachers themselves. This kind of assistance is perfectly defensible. Algebra abounds in investigations requiring the exercise of calm consideration, and of undisturbed and undivided attention. These cannot always be brought into operation amidst the multifarious duties of the school-room. What might be readily disposed of in the privacy of the closet, might occupy a good deal of time *there;* and among so many claimants for a portion of it, *time* is an object. The author, therefore, thinks it anything but discreditable in a teacher to refer now and then to a Key. It is unreasonable to expect that a difficulty, that may have detained an intelligent learner for an hour, though his best energies may have been concentrated upon it, should be removed by the teacher in a moment; and the author candidly confesses, that he should at once decline the challenge to solve even some of his own questions in this off-hand kind of way. He considers it, indeed, to be a perversion of the main purpose of mathematical study, to make it subservient to the mere knack of rapidly solving questions: a question that does not give exercise to thought and reflection is not worth the ink consumed in its solution. On this account a Key should be but sparingly used, even by the solitary student; while to the properly qualified teacher, its chief use is to refresh his memory as to the expedients and artifices best adapted to particular cases, and to save his time in the examination of the petty details and mere mechanical operations in the learner's work.

## ADDITION.—PAGE 22.

[Ex. 1.] The coefficients of $x$, in this example, when connected together by their proper signs, are $3-5+7+2-6+1-4$; the sum of the *positive* coefficients is evidently 13, and the sum of the negative coefficients, $-15$; and $13-15$ is $-2$: the first column of quantities therefore amounts to $-2x$. The coefficients in the second column are $2-3+1+4-5-7-6$; that is, $7-21$, or $-14$: therefore the second column amounts to $-14ab$; so that the sum of both columns is $-2x-14ab$.

[2.] The coefficients in the first column are $17-9+8+5-11+7$; that is, $37-20$, or 17. The coefficients in the second column are $-9-2+3-1+4-5$; that is, $7-17$, or $-10$: therefore the amount is $17xy-10cz$.

[3.] The coefficients of $x$ are $4+7-3-5+6$; that is, $17-8$, or 9: the coefficients of $y$ are $-2+5-3-4+6+1+7$; that is, $19-9$, or 10: the coefficients of $z$ are $3-1+4-2-5+6$; that is, $13-8$, or 5: lastly, the coefficients of $a$ are $-1+2+4-3+5-6$; that is, $11-10$, or 1: therefore the amount is $9x+10y+5z+a$, the coefficient 1 being omitted before the $a$, because *once* $a$ is $a$.

[4.] The coefficients of $axy$ are $1-3+2+12+9-7-8$; that is, $24-18$, or 6: the coefficients of $bz$ are $-3-2+4-5+1-3+7$; that is, $12-13$, or $-1$; and the numbers in the third column amount to $20-24$, that is to $-4$: therefore the whole result is $6axy-bz-4$.

[5.] The coefficients of $abcx$ are $-13+5-2-9+17+1-2$; that is, $23-26=-3$; the coefficients of $my$ are $2-4+3-7-9+5=10-20=-10$; and the coefficients of $nz$ are $-3+7-8-2+6+14=27-13=14$: therefore the amount is $-3abcx-10my+14nz$.

[6.] $1-7-3+12-2=13-12=1$, the coef. of $xy$; $-1-7+5-9-3+4+6=15-20=-5$, the coef. of $az$; $1+4-3+5-9-2+\frac{1}{2}=10\frac{1}{2}-14=-3\frac{1}{2}$, the coef. of $bc$; $-1+2-5+\frac{1}{2}+7-3-1=9\frac{1}{2}-10=-\frac{1}{2}$, the coef. of $d$: therefore the amount is $xy-5az-3\frac{1}{2}bc-\frac{1}{2}d$.

[7.] $2-1+6-3-5+2=1$, the coef. of $abx$; $-3+6-2+8-1=8$, the coef. of $dy$; $5-2+7-8-3-2+1=-2$, the coef. of $ez$; also the amount of the numbers in the last column is $-4$: therefore the whole amount is $abx+8dy-2ez-4$.

[8.] $4+5-7+9=11$, the coef. of $abx$; $-3+2+4-5=-2$, the coef. of $cdy$; $2-4-6-1=-9$, the coef. of $z$: therefore the amount is $11abx-2cdy-9z$.

[9.] $15-11-1=3$, the coef. of $mxy$; $7-13+1=-5$, the coef. of $nz$; $-3+1-2-25=-29$, the coef. of $ab$: therefore the amount is $3mxy-5nz-29ab$.

[10.] $10+3-7+1=7$, the coef. of $xyz$: $-3-5-2=-10$, the coef. of $abc$; $5-3+1=3$, the coef. of $de$: therefore the amount is $7xyz-10abc+3de$.

[11.] $-2+6-9=-5$, the coef. of $cx$; $3-7+1+4=1$, the coef. of $dy$; $4-5=-1$, the coef. of $xz$; $-8+14-6-8=-8$: therefore the amount is $-5cx+dy-xz-8$.

[12.] $5+7+3-1=14$, the coef. of $x$; $4-9+1=-4$, the coef. of $aby$; $2-1+3=4$, the coef. of $cz$; $7-12=-5$: therefore the amount is $14x-4aby+4cz-5$.

[13.] $-4+6+3-2+7-1-11+1=-1$, the coef. of $yz$; $2+5-13-1+15+5=13$, the coef. of $ax$; $-2-7+6+19-1+3-4=14$, the coef. of $bc$: therefore the amount is $-yz+13ax+14bc$, or $13ax-yz+14bc$.

[14.] $3+4-5+2-5=-1$, the coef. of $abx$; $-7-8+8+2+4=-1$, the coef. of $dy$; $1-3+6-1-3+2+4=6$, the coef. of $ez$: therefore the amount is $-abx-dy+6ez$.

---

## PAGE 25.

[1.] The coefficients of $ax$ are $3-1+2-8+1=-3$; those of $by$ are $-7-3+9-5+3=-3$; and those of $cz$, $2+4-1-4+11-5=7$: therefore the amount is $-3ax-3by+7cz+4d+5e$.

[2.] The coefficients of $abc$ are $6-4-3+1=0$, consequently there is *no* term like $abc$ in the result; the coefficients of $de$ are $-4+1+2-3=-4$; the coefficient of $m$ is 1, and those of $n$, $8-3=5$; also the numbers are $17+8+12-7-13-1=16$: therefore the amount is $-4de+m+5n+16$.

[3.] The coefficients of $xy$ are $4-2-1=1$; of $ab$, $-3-3+1=-5$; of $x$, $1-7-5+2=-9$; of $y$, $6+5+4=15$; of $a$, $8-5=3$; also $14-17=-3$: therefore the amount is $xy-5ab-9x+15y+3a-3$.

[4.] The coefficients of $y$ are $5+4+6=15$; of $b$, $-9$; of $x$, $-7+1-5+2=-9$; of $a$, $8-5=3$; of $xy$, $-2-1+4=1$; of $ab$, $1-3-3=-5$; also $14-17=-3$: therefore the amount is $15y-9b-9x+3a+xy-5ab-3$.

[5.] The coefficients of $xyz$ are $3-6-\frac{1}{2}-1=-4\frac{1}{2}$; of $xy$, $-2-\frac{1}{2}-2=-4\frac{1}{2}$; of $yz$, $5-1+7+4=15$; of $xz$, $\frac{1}{2}+7-3=4\frac{1}{2}$; of $x$, $2\frac{1}{2}+1=3\frac{1}{2}$: therefore the amount is $-4\frac{1}{2}xyz-4\frac{1}{2}xy+15yz+4\frac{1}{2}xz+3\frac{1}{2}x+3y+18$.

[6.] The coefficients of $abx$ are $7+7+1=15$; of $aby$, $-9-9-1=-19$; of $az$, $1+4-4=1$; of $ax$, $-6-5-1=-12$; of $ay$, $-12-3=\ 15$; of $abz$, $3+2-1=4$: therefore the amount is $15abx-19aby+az-12ax-15ay+4abz+x$.

[7.] The coefficients of $axy$ are $4-13+2+2=-5$; of $bxz$, $-2-6+7=-1$; of $c$, $3+7-1-5+6=10$; of $ab$, $5+1=6$: therefore the amount is $-5axy-bxz+10c+6ab-4d$.

[8.] The coefficients of $px$ are $2-5-4+1=-6$; of $qy$, $-3-4-7=-14$; of $rs$, $1+6-8=-1$; of $az$, $-1+5+2=6$; also the numbers are $-11+3+17=9$: therefore, including the quantity $a+3p-4q$, the amount is $-6px-14qy-rs+6az+a+3p-4q+9$.

[9.] The coefficients of $axy$ are $4-5+3=2$; of $bz$, $-7+2+6=1$; of $c$, $3+8-1-7=3$; of $x$, $5+3-7+9=10$; of $y$, $-7+4-13=-16$: therefore the amount is $2axy+bz+3c+10x-16y-4ab+1$.

## PAGE 32.

[1.] Collecting the coefficients of the like terms, we have—
$(3-2+5+1-4)(a+x)+(6+2-4+3-7)(b-y)=3(a+x)$.

[2.] $(4+6+5-6-2)\sqrt{a+x}+(2-3-3+2+4)\sqrt{b+y}= 7\sqrt{a+x}+2\sqrt{b+y}$.

[3.] $$(a+b+d+6+1)x^2y^3+(b-a-2+c-4)x^3y^2+(2-3-3-7+1)\sqrt{2}=$$
$$(a+b+d+7)x^2y^3+(b+c-a-6)x^3y^2-10\sqrt{2}.$$

[4.] $(ax-3x-3-4x)\sqrt{y}+(5y-by+2-2y)\sqrt{x}+6= (ax-7x-3)\sqrt{y}+(3y-by+2)\sqrt{x}+6$. But $ax-7x= (a-7)x$, and $3y-by=(3-b)y$: therefore the result is
$$\{(a-7)x-3\}\sqrt{y}+\{(3-b)y+2\}\sqrt{x}+6.$$

This form of the result, though the same in meaning as that which precedes, is to be preferred, because, when the letters are interpreted by *numbers*, the arithmetical work is rendered shorter and easier. Suppose, for instance, that $a$ were 13, and $x$ 25, then the work of $ax-7x$ would be $13\times 25-7\times 25$: but the work of $(a-7)x$ is only $6\times 25$. In like manner, if $b$ were 14, and $y$ 16, the work of $3y-by$ would be $3\times 16-14\times 16$; whereas the work of $(3-b)y$ is only $-11\times 16$. The learner may thus perceive, that not only is a needless repetition of symbols avoided by the use of brackets, as noticed at page 30 of the book, but that arithmetical facility is also promoted when the letters are translated into numbers.

(5.) $$(a+b+2+b+5)\sqrt{x}+(a+b-3+3a-b)xy+\sqrt{3}=$$
$$(a+2b+7)\sqrt{x}+(4a-3)xy+\sqrt{3}.$$

(6.) $$(5+a+2a+a+b)\sqrt{x+y}-(3+b+2b)\sqrt{x-y}=$$
$$(4a+b+5)\sqrt{x+y}-3(b+1)\sqrt{x-y}.$$

(7.) $$(1-2)\sqrt[4]{(x^2+1)}-(a-3+1-a-b)\sqrt{x}+(b-4+c+1)y$$
$$=-\sqrt[4]{(x^2+1)}+(b+2)\sqrt{x}+(b+c-3)y.$$

The learner will observe here that as the coefficients of $\sqrt{x}$ are put in brackets, preceded by a *minus* sign, the individual coefficients themselves are introduced with *changed signs*, agreeably to art. 23, page 33.

[8.] $(\sqrt{a+b}+\sqrt{a}+\sqrt{b}+\sqrt{ab})\sqrt[3]{(x+y)}+(3-b+c)xy.$

[9.] $(ab+a+b+\frac{a}{b})(x^2-y^2)+(a+b-ab+a-b)xy=$

$$(ab+a+b+\frac{a}{b})(x^2-y^2)+(2a-ab)xy=$$

$$\{(a+1)b+a(1+\frac{1}{b})\}(x^2-y^2)+a(2a-b)xy.$$

This last form is got from the preceding by putting $(a+1)b$ for its equal $ab+b$; $a(1+\frac{1}{b})$, for $a+\frac{a}{b}$; and $a(2a-b)$ for $2a-ab$. (See the remark on Ex. 4.)

[10.] $(1+a+b+a-a+b+3)(x+y)+(6+c-d-d+1)z-4=(a+2b+4)(x+y)+(c-2d+7)z-4.$

[11.] $(a+b-c+2a+3c-4+2c)x+(4-d+3+2)y-6-1=(3a+b+4c-4)x+(9-d)y-7.$

[12.] $(a^2-b^2-b^2+a^2+3a^2-3b^2)(x^2-y^2)+(3+5-1)xy=(5a^2-5b^2)(x^2-y^2)+7xy=5(a^2-b^2)(x^2-y^2)+7xy.$

[13.] $(m+4+2-3m)\sqrt{x}-(n-3+n-\frac{1}{2})\sqrt{y}+$

$$(1+3-a+b)\sqrt{z}=$$

$$2(3-m)\sqrt{x}-(2n-3\tfrac{1}{2})\sqrt{y}+(4-a+b)\sqrt{z}.$$

[14.] $(2\sqrt{a+b}+1)\sqrt[3]{x-y}-(2+a-2-3+2a)(x-y)=(2\sqrt{a+b}+1)\sqrt[3]{x-y}-3(a-1)(x-y).$ $[-3(a-1)=3(1-a)]$.

[15.] $a-b^2+c^2-d+e+3a+2b-c^2+d+e+5c^2+e-d+5b^2+4a+2b^2+c^2-4a-3e+6d=a+3a+4a-4a+2b-b^2+5b^2+2b^2+c^2-c^2+5c^2+c^2-d+d-d+6d+e+e+e-3e=4a+2b+6b^2+6c^2+5d.$

[16.] $a-b+2c+d-\{3a-4b+c-d\}+5\{2a-d+3a-c\}+2\{a-3b+3c\}=a-3a+25a+2a-b+4b-6b+2c-c-5c+6c+d+d-5d=25a-3b+2c-3d.$

## SUBTRACTION.—PAGE 37.

[1.] The coefficients of the lower row, when the signs are changed, are $2a-4b+1$; *adding* these to the coefficients above, we have $7a-7b+10$ for the coefficients of the remainder; so that the complete remainder is $7ax-7by+10z$.

[2.] The upper row is $-12xy+3a+3x-7$; the lower, with changed signs, is $10xy+4a-4x-8$: therefore, adding these, we have for remr. $-2xy+7a-x-15$.

[3.] The upper row is $8(a+x)-9(y-b)+c$; the lower row is $-8(a+x)+3(y-b)-a+c$: and *conceiving* the signs in this row to be changed, we have by addition the remainder, $16(a+x)-12(y-b)+a$.

[4.] The upper row is $3(x-y)+7\sqrt{x}-2\sqrt{y}$, and the lower, $6(x-y)-4\sqrt{x}+3\sqrt{y}$: therefore, changing signs, and adding $-3(x-y)-11\sqrt{x}+5\sqrt{y}=$ remr.

[5.] The upper row is $(-4+a)x^2+(2-b)y^2-\sqrt{z}$; and the lower, $(-a-6)x^2+(b-3)y^2+7$; changing signs, and adding $2(a+1)x^2+(5-2b)y^2-\sqrt{z}-7$.

[6.] When the coefficients are at all complicated, the learner will find it the easier plan to remove the brackets, and to attend to the coefficients *only:* thus

[6.]

$$\begin{array}{lrrr}
\text{coefs.} & a+b, & -b+c, & +3 \\
\text{coefs.} & 2a-3b, & +b+2c, & -a+3 \\
\hline
\text{coefs.} & -a+4b, & -2b-c, & +a.
\end{array}$$

[7.]

$$\begin{array}{rrr}
a-b+c, & -c+d, & +5 \\
3a+4b-c, & +3d, & -a+6 \\
\hline
-2a-5b+2c, & -c-2d, & a-1.
\end{array}$$

These results are the coefficients in the answer. The learner will observe that, in writing a compound coefficient, a *plus* term, if there be one, is always placed *first*, as a *sign* is thereby saved. Thus in Ex. 6, $(4b-a)v$, is a better form than $(-a+4b)v$, or $-(a-4b)v$, though each means the same thing.

[8.]

$$\begin{array}{rrrr}
a, & -b+c, & +c-b, & -3 \\
a-2b, & -2c+d, & +b-c, & -3+a \\
\hline
2b, & 3c-b-d, & 2c-2b, & -a.
\end{array}$$

[9.]

$$\begin{array}{rrrr}
2, & -3, & +4, & -2 \\
3b-6, & +4-a, & -7, & -17 \\
\hline
-3b+8, & -7+a, & 11, & 15.
\end{array}$$

[10.]

$$\begin{array}{rrrr} & 3a-3b, & -c-d & \\ & -5a+15b, & -2c+d, & +1 \\ \hline \text{coefs.} & 8a-18b, & c-2d, & -1. \\ \hline \end{array}$$

[11.]

$$\begin{array}{rrrr} a-3b, & -a+2b, & +1 & \\ b-2a, & b-3a, & 0, & +4 \\ \hline 3a-4b, & 2a+b, & 1, & -4. \\ \hline \end{array}$$

[12.]

$$\begin{array}{rrr} b+5c, & -a+\frac{1}{2}b, & -13 \\ 5c, & 2b-3a, & -a+1 \\ \hline b, & 2a-1\frac{1}{2}b, & a-14. \\ \hline \end{array}$$

[13.]

$$\begin{array}{rr} 2a-3c+b, & -d+e \\ 3c-4a+1\frac{1}{2}b, & -8 \\ \hline 6a-6c-\frac{1}{2}b, & -d+e+8. \\ \hline \end{array}$$

[14.]

$$\begin{array}{rr} 5a, & -c+4a \\ a-b, & 2a+3c \\ \hline 4a+b, & 2a-4c. \\ \hline \end{array}$$

[15.]

$$\begin{array}{rr} 4a-3b+5a+2c= & 9a-3b+2c \\ -2a+3a-4b-c= & a-4b-c \\ \cline{2-2} & 8a+b+3c. \\ \cline{2-2} \end{array}$$

[16.]

$$\begin{array}{rr} 2b-a-3b+c= & -b-a+c \\ 3c-b-2a+2c-3b= & -4b-2a+5c \\ \cline{2-2} & 3b+a-4c. \\ \cline{2-2} \end{array}$$

[17.]

$$\begin{array}{rr} & -a+2b-3c \\ 4c-5b-6+3a-2c= & 3a-5b+2c-6 \\ \cline{2-2} & -4a+7b-5c+6. \\ \cline{2-2} \end{array}$$

[18.]

$$\begin{array}{rr} d-2a-2c+5b-3d= & -2a+5b-2c-2d \\ & -3a+2b+1\frac{1}{2}c+7d \\ \cline{2-2} & a+3b-3\frac{1}{2}c-9d. \\ \cline{2-2} \end{array}$$

[19.]

$$\begin{array}{rr} x^2-10+4a-3b+5= & x^2+4a-3b-5 \\ 13b-2a+8-2x^2+2b-7= & -2x^2-2a+15b+1 \\ \cline{2-2} & 3x^2+6a-18b-6. \\ \cline{2-2} \end{array}$$

[20.]

$$\begin{array}{r}
-4a+7b-2c-5b+c=-4a+2b-c\\
6a-2c+3b+5a+c-b=\underline{\ \ 11a+2b-c}\\
\underline{-15a.\qquad\quad}
\end{array}$$

---

## MULTIPLICATION.—Case I. Pages 40, 41.

In these examples, the answers at the end of the book sufficiently explain the operation, with the exception, perhaps, of Ex. 11 and 12. The work of these is therefore here given.

$$\begin{array}{rr}
[11.]\ -2dxy^2\times a^2cz^2\times -7abd^3\times 3x^2yz= & 42a^3bcd^4x^3y^3z^3\\
-(2x-5x+6x)yz^2= & \underline{-3xyz^2\qquad\ }\\
\text{Product,} & \underline{-126a^3bcd^4x^4y^4z^5.}
\end{array}$$

$$\begin{array}{rr}
[12.]\ 5ab^2xy^3\times -2ax^4\times \tfrac{1}{2}x^2z^2=- & 5a^2b^2x^7y^3z^2\\
-3\,(a-2a-5a)\,xyz\times -ab=- & \underline{18a^2bxyz.\quad}\\
\text{Product,} & \underline{90a^4b^3x^8y^4z^3.}
\end{array}$$

---

### Case II. Page 42.

[1.]

$$\begin{array}{l}
2ab^2x-3x^2y\\
\underline{5a^2xy^2\qquad\qquad\qquad\quad}\\
\underline{10a^3b^2x^2y^2-15a^2x^3y^3.}
\end{array}$$

[2.]

$$\begin{array}{l}
\ \ 6b^2xy^2+axy-3bz^2\\
\underline{-5abx^2y^2\qquad\qquad\qquad\qquad\qquad\qquad\quad}\\
\underline{-30ab^3x^3y^4-5a^2bx^3y^3+15ab^2x^2y^2z^2.}
\end{array}$$

[3.]

$$\begin{array}{l}
-3xy-5az^2-b\\
\underline{-4ax^2y\qquad\qquad\qquad\qquad\qquad\quad}\\
\underline{12ax^3y^2+20a^2x^2yz^2+4abx^2y.}
\end{array}$$

[4.]

$$\begin{array}{l}
3x^2y^2z-2ax^3z^2-4\\
\underline{7ab^2x\qquad\qquad\qquad\qquad\qquad\qquad\quad}\\
\underline{21ab^2x^3y^2z-14a^2b^2x^4z^2-28ab^2x.}
\end{array}$$

[5.]

$$\begin{array}{l}
-2ax+4c^2y+abc^2\\
\underline{\ \ 6c^2xz^2\qquad\qquad\qquad\qquad\qquad\qquad\quad}\\
\underline{-12ac^2x^2z^2+24c^4xyz^2+6abc^4xz^2.}
\end{array}$$

[6.]

$$\begin{array}{l}
-2xy+3ax--a^3\\
\underline{-5a^2y^3z^4\qquad\qquad\qquad\qquad\qquad\quad}\\
\underline{10a^2xy^4z^4-15a^3xy^3z^4+5a^5y^3z^4.}
\end{array}$$

[7.]

$$-\ ax-y^2+3a^3$$
$$2a^2xy\times-3ay^2=-6a^3xy^3$$
$$\underline{6a^4x^2y^3+6a^3xy^5-18a^6xy^3.}$$

[8.]

$$2axy-3b^2-4$$
$$7a^2bx^2z^3$$
$$\underline{14a^3bx^3yz^3-21a^2b^3x^2z^3-28a^2bx^2z^3.}$$

[9.]

$$3c^2-2ab^2+1$$
$$-5abc\times-4b^2c^3\times-6a^2b=-120a^3b^4c^4$$
$$\underline{-360a^3b^4c^6+240a^4b^6c^4-120a^3b^4c^4.}$$

[10.]

$$-3a^2x^2+a^3-9$$
$$-2x^2y^5$$
$$\underline{6a^2x^4y^5-2a^3x^2y^5+18x^2y^5.}$$

[11.]

$$x^3-3ax^2+bx-c$$
$$-a^2b^3c^4$$
$$\underline{-a^2b^3c^4x^3+3a^3b^3c^4x^2-a^2b^4c^4x+a^2b^3c^5.}$$

[12.]

$$-4ax^2y+3yz^2+8$$
$$-3xyz^3$$
$$\underline{12ax^3y^2z^3-9xy^2z^5-24xyz^3.}$$

[13.]

$$ax-2by+3a^2x^2$$
$$-4abxy$$
$$\underline{-4a^2bx^2y+8ab^2xy^2-12a^3bx^3y.}$$

[14.]

$$-\ b^2x^2y-cxy^2+2$$
$$-5x^2yz$$
$$\underline{5b^2x^4y^2z+5cx^3y^3z-10x^2yz.}$$

[15.]

$$5x^2z+2aby-3a^2b^2x^2+4bxy^2+2az^2-3$$
$$-\ 2a^3b^2xyz^2.$$
$$-10a^3b^2x^3yz^3-4a^4b^3xy^2z^2+6a^5b^4x^3yz^2-8a^3b^3x^2y^3z^2-$$
$$\underline{4a^4b^2xyz^4+6a^3b^2xyz^2.}$$

[16.] $9+1-8+4-9+25=22$; and $-a^2bx\times 22=-22a^2bx$.

[17.] $16-4-27-7+16=-6$; and $5-49+3=-41$; and $-6\times -41\times x^2y^2=246x^2y^2$.

[18.] $1-8-3+25=15$; and $3-8+16=11$; and $15abx\times -11b^2z=-165ab^3xz$.

[19.] $4-(3-9)ax+1=5+6ax$

$$\begin{array}{l}
5+6ax\\
-5a^2xy^3\\
\hline
-25a^2xy^3-30a^3x^2y^3.\\
\hline
\end{array}$$

[20.]

$$\begin{array}{l}
3a^2y-2bx^2+3y^2+b^3\\
-2a^3xy^2\\
\hline
-6a^5xy^3+4a^3bx^3y^2-6a^3xy^4-2a^3b^3xy^2.\\
\hline
\end{array}$$

## CASE III. PAGE 45.

[1.]

$$\begin{array}{rrrr}
4x^2 & -3x & +5 & \\
2x^2 & -x & & \\
\hline
8x^4 & -6x^3 & +10x^2 & \\
 & -4x^3 & +3x^2 & -5x\\
\hline
8x^4 & -10x^3 & +13x^2 & -5x.\\
\hline
\end{array}$$

[2.]

$$\begin{array}{rrrrr}
3x^2 & +6x & -4 & & \\
x^2 & -3x & +2 & & \\
\hline
3x^4 & +6x^3 & -4x^2 & & \\
 & -9x^3 & -18x^2 & +12x & \\
 & & 6x^2 & +12x & -8\\
\hline
3x^4 & -3x^3 & -16x^2 & +24x & -8\\
\hline
\end{array}$$

[3.]

$$\begin{array}{rrrrrr}
x^3 & +2x^2 & -3x & +4 & & \\
x^2 & -x & -3 & & & \\
\hline
x^5 & +2x^4 & -3x^3 & +4x^2 & & \\
 & -x^4 & -2x^3 & +3x^2 & -4x & \\
 & & -3x^3 & -6x^2 & +9x & -12\\
\hline
x^5 & +x^4 & -8x^3 & +x^2 & +5x & -12.\\
\hline
\end{array}$$

[4.]

$$\begin{array}{l}
ax^2+bx-c\\
dx-e\\
\hline
adx^3+bdx^2-cdx\\
\qquad\quad -aex^2-bex+ce\\
\hline
adx^3+(bd-ae)x^2-(be+cd)x+ce.\\
\hline
\end{array}$$

[5.]

$$\begin{array}{l}
x^3 - ax^2 + bx + c \\
x^2 - ex + f \\
\hline
x^5 - ax^4 + bx^3 + cx^2 \\
\quad - ex^4 + aex^3 - bex^2 - cex \\
\qquad\qquad fx^3 - afx^2 + bfx + cf \\
\hline
x^5 - (a+e)x^4 + (b+ae+f)x^3 + (c-be-af)x^2 + (bf-ce)x + cf. \\
\hline
\end{array}$$

[6.]

$$\begin{array}{l}
x^2 + (a-b)x + c \\
mx + n - 4 \\
\hline
mx^3 + m(a-b)x^2 + cmx \\
\qquad\qquad (n-4)x^2 + (a-b)(n-4)x + c(n-4) \\
\hline
mx^3 + \{m(a-b) + n - 4\}x^2 + \{(a-b)(n-4) + cm\}x + c(n-4). \\
\hline
\end{array}$$

[7.]

$$\begin{array}{l}
x^3 + x^2y + xy^2 + y^3 \\
x - y \\
\hline
x^4 + x^3y + x^2y^2 + xy^3 \\
\quad - x^3y - x^2y^2 - xy^3 - y^4 \\
\hline
\qquad x^4 - y^4 \\
\hline
\end{array}$$

[8.]

$$\begin{array}{l}
a^4 - a^3x + a^2x^2 - ax^3 + x^4 \\
a + x \\
\hline
a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 \\
\qquad a^4x - a^3x^2 + a^2x^3 - ax^4 + x^5. \\
\hline
\qquad a^5 + x^5 \\
\hline
\end{array}$$

[9.]

$$\begin{array}{rl}
 & x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 \\
(x+y)^2 = & x^2 + 2xy + y^2 \\
\hline
 & x^6 - 2x^5y + 3x^4y^2 - 2x^3y^3 + x^2y^4 \\
 & \qquad 2x^5y - 4x^4y^2 + 6x^3y^3 - 4x^2y^4 + 2xy^5 \\
 & \qquad\qquad x^4y^2 - 2x^3y^3 + 3x^2y^4 - 2xy^5 + y^6 \\
\hline
 & x^6 \qquad\qquad + 2x^3y^3 \qquad\qquad + y^6 = (x^3 + y^3)^2. \\
\hline
\end{array}$$

[10.]

$(a+b)(a-b) = a^2 - b^2$; also, p. 44, $(a^2+b^2)(a^2-b^2) = a^4 - b^4$; and $(a-b)^2 = a^2 - 2ab + b^2$ [See II. p. 44.]

$$\begin{array}{rl}
 & a^4 - b^4 \\
\hline
 & a^6 - 2a^5b + a^4b^2 \\
 & \qquad - a^2b^4 + 2ab^5 - b^6 \\
\hline
\text{therefore, } (a^4 - b^4)(a-b)^2 = & a^6 - 2a^5b + a^4b^2 - a^2b^4 + 2ab^5 - b^6. \\
\hline
\end{array}$$

This example brings into application the principle mentioned at page 44; namely, that the sum multiplied by the difference of two quantities gives the difference of their squares. The operation above, like most of those here exhibited, is intended chiefly for practice in the multiplication of algebraic quantities; but the learner is not to suppose that the products here obtained are always in a form best suited for arithmetical computation, when the letters are replaced by numbers. On the contrary, it generally happens that, for this purpose, the factors of the product are more eligible than the developed product itself, though such is not invariably the case. Thus, in the present example, the form—

$$(a-b)^2(a+b)(a-b)(a^2+b^2) \text{ or } (a-b)^3(a+b)(a^2+b^2)\text{—}$$

would be preferred, for arithmetical calculation, to the worked out product given above; and, in general, it is of more importance to replace a product by its factors, than by multiplying the factors together to exhibit the product. Sometimes, however, the actual multiplication is of practical advantage, as brevity and simplicity is secured by it. Examples 7, 8, and 9, are instances of this, as the learner will readily see from the form of the results.

[11.] By theorem 3, page 44, the value of $(x-7)(x+5)$ is found, without the trouble of actual multiplication, to be $x^2-2x-35$; and the value of $(x+2)(x+6)$ to be $x^2+8x+12$: the operation is therefore as follows:—

$$\begin{array}{lllll}
x^2-2x-35 & & & & \\
\underline{x^2+8x+12} & & & & \\
x^4-2x^3 & -35x^2 & & & \\
\phantom{x^4}+8x^3 & -16x^2 & -280x & & \\
 & +12x^2 & -24x & -420 & \\
\hline
x^4+6x^3 & -39x^2 & -304x & -420. &
\end{array}$$

The learner is strongly recommended to familiarise himself with the theorem here brought into operation; it will often save him the trouble of petty multiplications. Indeed, the *three* theorems at page 44 should all be kept in remembrance, so as to be ready for use whenever occasion requires.

[12.] In this example, the expressions to be multiplied together are found by the theorems at page 44 to be, $x^2 + 4x + 4$, $x^2 + 4x + 4$, and $x^2 - 7x + 12$: therefore the work is as follows:—

$$
\begin{array}{lllllll}
x^2 & +4x & +4 & & & & \\
x^2 & +4x & +4 & & & & \\
\hline
x^4 & +4x^3 & +4x^2 & & & & \\
 & 4x^3 & +16x^2 & +16x & & & \\
 & & 4x^2 & +16x & +16 & & \\
\hline
x^4 & +8x^3 & +24x^2 & +32x & +16 & & \\
x^2 & -7x & +12 & & & & \\
\hline
x^6 & +8x^5 & +24x^4 & +\ 32x^3 & +\ 16x^2 & & \\
 & -7x^5 & -56x^4 & -168x^3 & -224x^2 & -112x & \\
 & & 12x^4 & +\ 96x^3 & +288x^2 & +384x & +192 \\
\hline
x^6 & +x^5 & -20x^4 & -\ 40x^3 & +\ 80x^2 & +272x & +192. \\
\hline
\end{array}
$$

When compound expressions are to be multiplied together, and they are such that the leading term in each contains a power of some letter, the next term a power differing from the former by 1, the next a power differing by 2, and so on—that is, when the powers regularly descend or ascend, as in the present example—the work may be abridged by omitting these powers altogether, retaining only the *coefficients* of them. Thus, in each of the first two factors above, the exponent of $x$ in the leading term is 2; in the next term, 1; and in the third term, $x$ does not enter at all: the powers of $x$ therefore regularly descend; also in the expression $x^4 + 8x^3 + 24x^2 + 32x + 16$, the exponents of $x$ descend in like manner regularly, each being a unit less than the preceding one. It is plain, in multiplying two such expressions together, that the exponent in the leading term of the product will be the *sum* of the exponents in the leading term of each factor; and that the succeeding exponents will, in like manner, regularly descend: if therefore we knew only the *coefficients* of the product, the necessary *powers* of the letter to which they belong could easily be supplied: in long multiplications, therefore, the letter spoken of may always be suppressed till the close of the operation with the coefficients. Adopting this abridged plan, the above example may be worked thus:—

$$\begin{array}{rrrrrrr}
1 & +4 & +4 & & & & \\
1 & +4 & +4 & & & & \\
\hline
1 & +4 & +4 & & & & \\
 & 4 & +16 & +16 & & & \\
 & & 4 & +16 & +16 & & \\
\hline
1 & +8 & +24 & +32 & +16 & & \\
1 & -7 & +12 & & & & \\
\hline
1 & +8 & +24 & +32 & +16 & & \\
 & -7 & -56 & -168 & -224 & -112 & \\
 & & 12 & +96 & +288 & +384 & +192 \\
\hline
1 & +1 & -20 & -40 & +80 & +272 & +192.
\end{array}$$

This result supplies the *coefficients* of $x$ in the product; and as we know that the leading power of $x$ is $x^6$, and that the powers regularly descend, the proper powers may be at once introduced, and the complete product—

$$x^6 + x^5 - 20x^4 - 40x^3 + 80x^2 + 272x + 192$$

written down.

The same abridged form may be adopted when the powers of $x$ do not regularly descend or ascend; because we can always bring about uniformity in this respect by filling up the gaps with the wanting powers, written with *zero-coefficients.* Thus, such an expression as $x^5 - 3x^3 + 5x - 2$ might be written $x^5 + 0x^4 - 3x^3 + 0x^2 + 5x - 2$; and any two such expressions, the gaps being filled up in this way, might be multiplied together, without $x$, as above. This is called *the method of detached coefficients.*

[13.] By theorem 1, page 44, $(x^2 + y^2)(x^2 - y^2) = x^4 - y^4$; and $(x^4 + y^4)(x^4 - y^4) = x^8 - y^8$.

[14.]

$$\begin{array}{rrrrr}
2x^2 & -\frac{1}{2}x & +2 & & \\
x^2 & -2x & +1 & & \\
\hline
2x^4 & -\frac{1}{2}x^3 & +2x^2 & & \\
 & -4x^3 & +x^2 & -4x & \\
 & & 2x^2 & -\frac{1}{2}x & +2 \\
\hline
2x^4 & -\frac{9}{2}x^3 & +5x^2 & -\frac{9}{2}x & +2;
\end{array}
\qquad \text{or,} \qquad
\begin{array}{rrrrr}
2 & -\frac{1}{2} & +2 & & \\
1 & -2 & +1 & & \\
\hline
2 & -\frac{1}{2} & +2 & & \\
 & -4 & +1 & -4 & \\
 & & 2 & -\frac{1}{2} & +2 \\
\hline
2 & -\frac{9}{2} & +5 & -\frac{9}{2} & +2:
\end{array}$$

therefore, supplying the powers of $x$, the result is the same as before.

[15.]

$$\begin{array}{l}
bx - ay \\
cx - by \\
\hline
bcx^2 - acxy \\
\qquad\quad - b^2xy + aby^2 \\
\hline
bcx^2 - (b^2 + ac)xy + aby^2 \\
\quad ax \; + by - cz \\
\hline
abcx^3 - (ab^2 + a^2c)x^2y + a^2bxy^2 \\
\qquad\qquad\qquad\qquad b^2c\; x^2y - (b^3 + abc)xy^2 + ab^2y^3.
\end{array}$$

To the sum of these two products must be added the product by $-cz$. It is plain, on account of the new letter $z$, that no term of the latter product will have its *like* among the terms of the products here exhibited; so that there would be no use in writing the terms of the final product underneath those already obtained, with a view to their *addition:* it will be sufficient to add up the terms already found, and then to annex the terms of the product by $-cz$: the complete product is thus:—

$$abcx^3 + (b^2c - a^2c - ab^2)x^2y + (a^2b - abc - b^3)xy^2 + ab^2y^3 - bc^2x^2z + (ac + b^2)cxyz - abcy^2z.$$

[16.]

$$\begin{array}{l}
x^2 - (a - b)x + c \\
x^2 - (c + d)x - ab \\
\hline
x^4 - (a - b)x^3 + cx^2 \\
\quad - (c + d)x^3 + (a - b)(c + d)x^2 - c(c + d)x \\
\qquad\qquad\qquad\qquad\qquad - abx^2 + ab(a - b)x - abc \\
\hline
x^4 + (b - a - c - d)x^3 + \{(a - b)(c + d) - ab + c\}x^2 + \{ab(a - b) - \\
\qquad\qquad\qquad\qquad c(c + d)\}x - abc. \\
\hline
\end{array}$$

[17.]

$$\begin{array}{l}
ax^2 + (a + b)x + c \\
bx^2 - (b - a)x - d \\
\hline
abx^4 + (ab + b^2)x^3 + bcx^2 \\
\quad - (ab - ac)x^3 - (ab + b^2 - ac - bc)x^2 - (b - c)cx \\
\qquad\qquad\qquad\qquad\qquad\qquad - adx^2 - (a + b)dx - cd \\
\hline
abx^4 + (ac + b^2)x^3 + \{(2c - b)b - a(b - c + d)\}x^2 - \{(b - c)c + \\
\qquad\qquad\qquad\qquad (a + b)d\}x - cd. \\
\hline
\end{array}$$

[18.]

$$\begin{array}{l}
2ax^3-(a-b)x^2+4 \\
3bx^3-ex^2+2a \\
\hline
6abx^6-3b(a-b)x^5+12bx^3 \\
\qquad\qquad\qquad -2aex^5+e(a-b)x^4-4ex^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad 4a^2x^3-2a(a-b)x^2+8a \\
\hline
6abx^6-\{3b(a-b)+2ae\}x^5+e(a-b)x^4+4(a^2+3b)x^3- \\
\qquad\qquad\qquad 2\{a(a-b)+2e\}x^2+8a \\
\hline
\end{array}$$

[19.] Using here only detached coefficients, as explained at page 15 of the Key, we proceed thus:—

$$\begin{array}{l}
-7a+3b-5c+6e-2 \\
\quad 4a-7b-5 \\
\hline
-28a^2+12ab-20ac+24ae-8a \\
\qquad\qquad 49ab-21b^2+35bc-42be+14b \\
\qquad\qquad\qquad\qquad 35a\ -15b\ +25c\ -30e+10 \\
\hline
-28a^2+61ab+\{5a(7-4c)-21b^2\}+\{24ae+5b(7c- \\
\qquad 3)\}+(25c-42be-8a)+(14b-30e)+10
\end{array}$$

These are the coefficients of the several powers of $x$ in the product, the first power being $x^6$; by introducing $x^6$, $x^5$, $x^4$, $x^3$, $x^2$, $x$, we shall have the result in the book.

[20.] By theorem 1, page 44, $\{x^2-(ax+b\}\{x^2+(ax+b)\}=x^4-(ax+b)^2=$ (theorem 2), $x^4-a^2x^2-2abx-b^2$: it remains therefore to multiply this by $x^2+ax-b$; and this we shall do by using detached coefficients, as in the preceding example. (See the remarks at page 15, Key.)

$$\begin{array}{l}
1+0-a^2-2ab-b^2 \\
1+a-b \\
\hline
1+0-a^2-2ab-b^2 \\
\quad a+0-a^3-2a^2b-ab^2 \\
\qquad -b+\ 0\ +\ a^2b+2ab^2+b^3 \\
\hline
1+a-(a^2+b)-a(a^2+2b)-b(+a^2+b)+ab^2+b^3. \\
\hline
\end{array}$$

By connecting $x^6$, $x^5$, $x^4$, $x^3$, $x^2$, $x$, with these several coefficients, the result in the book is obtained.

[21.] The first factor is $a^4+a^2x^2+x^4$, which is multiplied by $a^2-x^2$ as follows:—

$$\begin{array}{l} a^2-x^2 \\ \hline a^6+a^4x^2+a^2x^4 \\ \quad\; -a^4x^2-a^2x^4-x^6 \\ \hline \qquad a^6-x^6 \quad =(a^3+x^3)(a^3-x^3). \end{array}$$

[22.]

$$\begin{array}{l} x^2-(a+b)x\ +a(a-b)+b^2 \\ x\ +(a+b) \\ \hline x^3-(a+b)x^2+\{a(a-b)+b^2\}x \\ \qquad (a+b)x^2-\quad (a+b)^2x+a(a^2-b^2)+(a+b)b^2 \\ \hline x^3 \qquad\qquad\qquad -3abx\ \ +a^3 \qquad +b^3 \\ \hline \end{array}$$

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## DIVISION.—CASE I. PAGE 48.

[1.] $\frac{-14ax^5z^6}{2ax^3z^4}=-7x^2z^2.$ [2.] $\frac{12a^3b^2y^3z^2}{-3abx^2}=-\frac{4a^2by^3z^2}{x^2}.$

[3.] $\frac{5a^2y^5z^8}{a^2y^3z^4}=5y^2z^4.$ [4.] $\frac{-9b^3xz^3}{-3ab^3xy}=\frac{3z^3}{ay}.$

[5.] $\frac{3x^2y^2z^3}{-5x^3y^3z^3}=-\frac{3}{5xy}.$ [6.] $\frac{a^3bx^4y^5}{\frac{1}{2}axyz}=\frac{2a^2bx^3y^4}{z}.$

[7.] $\frac{4x^2y^m}{-7xy^n}=-\frac{4}{7}xy^{m-n}.$ [8.] $\frac{-3x^my^nz^4}{2x^3y^2z}=-\frac{3}{2}x^{m-3}y^{n-2}z^3.$

[9.] $\frac{5x^{\frac{1}{2}}yz^3}{-4ay^2\sqrt{x}}=-\frac{5z^3}{4ay}.$ [10.] $\frac{8a^{2n}x^{3m}y^5}{4a^nx^my^2}=2a^nx^{2m}y^3.$

[11.]

$\frac{-3ax^{m+1}y^{n-1}}{2ax^my^3}=-\frac{3}{2}xy^{n-4}.$

[12.]

$\frac{5x^{n-1}y^{m+2}}{3x^2\ y^{n-3}}=\frac{5}{3}x^{n-3}y^{m-n+5}.$

[13.]

$\frac{-b^3x^{3m}y^{2n+3}}{-3b^2x^{m+2}y^{n-3}}=\frac{1}{3}bx^{2m-2}y^{n+6}.$

[14.]

$\frac{15a^{3m}x^ny^{2m}}{9a^{m-3}x^4y^{m+2}}=\frac{5}{3}a^{2m+3}x^{n-4}y^{m-2}.$

## CASE II. PAGE 49.

[1.] $\frac{8a^4x^2y^3-4a^3x^2y^4-axy}{2axy}=4a^3xy^2-2a^2xy^3-\frac{1}{2}.$

[2.] $\frac{6x^3y^3z^2-3x^2y^2z^3-5xyz^4}{-3x^2yz^2}=-2xy^2+yz+\frac{5z^2}{3x}.$

[3.] $\frac{a^2b^3c^4-3a^3b^2-4b^4c^3}{3a^2bc^2}=\frac{1}{3}b^2c^2-\frac{ab}{c^2}-\frac{4b^3c}{3a^2}.$

[4.] $\frac{6ax^2y-12a^2x}{4xy}=\frac{3}{2}ax-\frac{3a^2}{y}.$

[5.]

$$\frac{(2ab^2-b)(3ax-x^2)}{2bx}=\frac{(2ab-1)(3a-x)bx}{2bx}=\frac{1}{2}(2ab-1)(3a-x).$$

[6.]

$$\frac{4a(3bc+d^2)(ab-d)}{3a^2b^2}=\frac{4(3ab^2c-3bcd+abd^2-d^3)}{3ab^2}=$$

$$4\left(c-\frac{cd}{ab}+\frac{d^2}{3b}-\frac{d^3}{3ab^2}\right).$$

[7.]

$$\frac{2ax^3(b-4cx)(ab+x)}{8ax^4}=\frac{(b-4cx)(ab+x)}{4x}=\frac{ab^2+(b-4abc)x-4cx^2}{4x}=$$

$$\frac{ab^2}{4x}+\frac{b}{4}-abc-cx.$$

[8.]

$$\frac{(a^2-x^2)(6ax+b)+bx^2}{3ax^2}=\frac{-6ax^3+6a^3x+a^2b}{3ax^2}=-2x+\frac{2a^2}{x}+\frac{ab}{3x^2}.$$

[9.] The dividend is $15a^3xy+10a^2by^2-(9b+4a)ax^2-4b^2xy$, and this, divided by $3a^3x$, gives

$$5y+\frac{10by^2}{3ax}-\frac{3bx}{a^2}-\frac{4x}{3a}-\frac{4b^2y}{3a^3}.$$

[10.] The dividend is $(x^2-2ax-24a^2)(x^2-3x+2a)=x^4-(2a+3)x^3+(8a-24a^2)x^2+68a^2x-48a^3$; and this, divided by $8ax^2$, gives

$$\frac{1}{8a}x^2-\frac{1}{4}x-\frac{3}{8a}x+1-3a+\frac{17a}{2x}-\frac{6a^2}{x^2}.$$

[11.] The dividend is $(4ax-4b^2+a)(3x^2-2ab^2)=12ax^3+(3a-12b^2)x^2-8a^2b^2x+8ab^4-2a^2b^2$, and this, divided by $-ab^3x^2$, gives

$$-\frac{12x}{b^3}-\frac{3}{b^3}+\frac{12}{ab}+\frac{8a}{bx}-\frac{8b}{x^2}+\frac{2a}{bx^2}.$$

[12.] Here we have only to subtract $n-2$ from each of the exponents, and to change the signs of the coefficients, to get the quotient as in the book.

[13.]
$$\begin{array}{l}
\phantom{-}3x^2+(a\ \ -2a^2) \\
\underline{-2x^2+(a^2+2a\ \ )} \\
-6x^4-2(a\ \ -2a^2)x^2 \\
\underline{\qquad\qquad 3(a^2+2a\ \ )x^2+a^3-2a^4+2a^2-4a^3} \\
\underline{-6x^4+7a^2x^2+4ax^2+2a^2-3a^3-2a^4}
\end{array}$$

Dividing this by $2a^2x^2$, the quotient is

$$-\frac{3x^2}{a^2}+\frac{7}{2}+\frac{2}{a}+\frac{1}{x^2}-\frac{3a}{2x^2}-\frac{a^2}{x^2},$$

and this multiplied by 3, the factor in the dividend which has been omitted, the result is the quotient required.

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## CASE III. PAGE 54.

[1.]
$$\begin{array}{l}
2x-5)6x^2-\ 5x-25(3x+5 \\
\qquad\quad \underline{6x^2-15x} \\
\qquad\qquad\quad 10x-25 \\
\qquad\qquad\quad \underline{10x-25}
\end{array}$$

[2.] $$\begin{array}{l} 3x-4)6x^3-23x^2+26x-8(2x^2-5x+2 \\ \underline{6x^3-\ \ 8x^2} \\ \quad -15x^2+26x \\ \quad \underline{-15x^2+20x} \\ \qquad\qquad 6x-8 \\ \qquad\qquad \underline{6x-8} \end{array}$$

[3.] $$\begin{array}{l} a-x)a^3-x^3(a^2+ax+x^2 \\ \underline{a^3-a^2x} \\ \quad a^2x-x^3 \\ \quad \underline{a^2x-ax^2} \\ \qquad ax^2-x^3 \\ \qquad \underline{ax^2-x^3} \end{array}$$

[4.]* $$\begin{array}{l} x^2-2ax+4a^2)x^4+4a^2x^2+16a^4(x^2+2ax+4a^2 \\ \underline{x^4-2ax^3+\ 4a^2x^2} \\ \quad 2ax^3\ +16a^4 \\ \quad \underline{2ax^3-\ 4a^2x^2+8a^3x} \\ \qquad\qquad 4a^2x^2-8a^3x+16a^4 \\ \qquad\qquad \underline{4a^2x^2-8a^3x+16a^4} \end{array}$$

[5.] $$\begin{array}{l} 4x^2-2xy+y^2)8x^3+y^3(2x+y \\ \underline{8x^3-4x^2y+2xy^2} \\ \quad 4x^2y-2xy^2+y^3 \\ \quad \underline{4x^2y-2xy^2+y^3} \end{array}$$

[6.] $$\begin{array}{l} x^2-xy+y^2)x^4+x^2y^2+y^4\ (x^2+xy+y^2 \\ \underline{x^4-x^3y+x^2y^2} \\ \quad x^3y\ \ +y^4 \\ \quad \underline{x^3y\ \ -x^2y^2+xy^3} \\ \qquad\qquad x^2y^2-xy^3+y^4 \\ \qquad\qquad \underline{x^2y^2-xy^3+y^4} \end{array}$$

The learner will observe that the quotient in this example might have been discovered, without actual division, by aid of the frequently-employed theorem marked 1, at page 44: for the dividend is evidently the same as $(x^2+y^2)^2-x^2y^2$; and, by the theorem referred to, we know that

---

* In examples [1.], [2.], and [4.], the work might have been conducted with *detached coefficients* only, in imitation of the multiplication operation given at page 16; observing, as in multiplication, to fill up the places where powers of $x$ are absent, by *zeros*; thus, the dividend in ex. 4, when the coefficients only are used, should be written $1+0+4a^2+0+16a^4$, the divisor being $1-2a+4a^2$: the learner is recommended to work the example in this form, and he will easily get the coefficients of the quotient; namely $1+2a+4a^2$.

$$\frac{(x^2+y^2)^2-x^2y^2}{x^2+y^2-xy}=x^2+y^2+xy$$

[7.]

$$\begin{array}{l}
x+2y)\,x^5+32y^5\,(x^4-2x^3y+4x^2y^2-8xy^3+16y^4 \\
\qquad\qquad \underline{x^5+\ 2x^4y} \\
\qquad\qquad\quad -\ 2x^4y+32y^5 \\
\qquad\qquad\quad \underline{-\ 2x^4y-\ 4x^3y^2} \\
\qquad\qquad\qquad\qquad\quad 4x^3y^2+32y^5 \\
\qquad\qquad\qquad\qquad\quad \underline{4x^3y^2+\ 8x^2y^3} \\
\qquad\qquad\qquad\qquad\qquad\quad -\ 8x^2y^3+32y^5 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{-\ 8x^2y^3-16xy^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 16xy^4+32y^5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{16xy^4+32y^5}
\end{array}$$

In this example the term $32y^5$ need not to have been brought down till the last term of the quotient had been obtained.

[8.]

$$\begin{array}{l}
3a^2-5ab+2b^2)\,3a^4-26a^3b+37a^2b^2-14ab^3\,(a^2-7ab \\
\qquad\qquad\qquad \underline{3a^4-\ 5a^3b+\ 2a^2b^2} \\
\qquad\qquad\qquad\qquad -21a^3b+35a^2b^2-14ab^3 \\
\qquad\qquad\qquad\qquad \underline{-21a^3b+35a^2b^2-14ab^3}
\end{array}$$

[9.]

$$\begin{array}{l}
2a^2-3ax+4x^2)\,10a^4-27a^3x+34a^2x^2-18ax^3-8x^4\,(5a^2-6ax-2x^2 \\
\qquad\qquad\qquad \underline{10a^4-15a^3x+20a^2x^2} \\
\qquad\qquad\qquad\qquad -12a^3x+14a^2x^2-18ax^3 \\
\qquad\qquad\qquad\qquad \underline{-12a^3x+18a^2x^2-24ax^3} \\
\qquad\qquad\qquad\qquad\qquad\quad -\ 4a^2x^2+\ 6ax^3-8x^4 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{-\ 4a^2x^2+\ 6ax^3-8x^4}
\end{array}$$

[10.]

$$3x^3+2x^2y^2+3y^3)\,6x^6-5x^5y^2-6x^4y^4+6x^3y^3+15x^3y^3-9x^2y^4+10x^2y^5+15y^5\,(2x^3-3x^2y^2+5y^3$$
$$\underline{6x^6+4x^5y^2+6x^3y^3}$$
$$-9x^5y^2-6x^4y^4+15x^3y^3$$
$$\underline{-9x^5y^2-6x^4y^4-\ 9x^2y^4}$$
$$15x^3y^3+10x^2y^5+15y^5$$
$$\underline{15x^3y^3+10x^2y^5+15y^5}$$

[11.] $$4y^2-2yz+z^2)\,16y^4-8y^3z+2yz^3-z^4\,(4y^2-z^2$$
$$\underline{16y^4-8y^3z+4y^2z^2}$$
$$-4y^2z^2+2yz^3-z^4$$
$$\underline{-4y^2z^2+2yz^3+z^4}$$

This example may be otherwise worked, by aid of theorem 1, page 44, thus—

$$(4y^2+z^2)-2yz\big)(16y^4-z^4)-2yz(4y^2-z^2)\big(4y^2-z^2$$
$$\underline{(16y^4-z^4)-2yz(4y^2-z^2)}$$

Here the compound term $(16y^4-z^4)$, in the dividend, is the *difference of two squares;* and the compound term in the divisor, $(4y^2+z^2)$, is the *sum* of the quantities thus squared: hence their *difference* $(4y^2-z^2)$ is suggested, as the leading compound term of the quotient, which is found to be the *complete* quotient.

[12.]

$$2x^2+3xy-y^2)\,4x^5-3x^3y^2+13x^2y^3-7xy^4+y^5\,(2x^3-3x^2y+4xy^2-y^3$$
$$\underline{4x^5+6x^4y-\ 2x^3y^2}$$
$$-6x^4y-\ x^3y^2+13x^2y^3$$
$$\underline{-6x^4y-\ 9x^3y^2+\ 3x^2y^3}$$
$$8x^3y^2+10x^2y^3-7xy^4$$
$$\underline{8x^3y^2+12x^2y^3-4xy^4}$$
$$-\ 2x^2y^3-3xy^4+y^5$$
$$\underline{-\ 2x^2y^3-3xy^4+y^5}$$

It is of importance, in division operations such as these, that the learner should always examine the several terms of the dividend *before* attempting the *subtraction*, in order that terms *like* those to be subtracted may not be overlooked: examples 4, 5, 10 and 12, show the necessity of this.

[13.] $x^3+4x^2y+2xy^2+3y^3)x^4+3x^3y-2x^2y^2+xy^3-3y^4(x-y$

$$\begin{array}{r}
\underline{x^4+4x^3y+2x^2y^2+3xy^3} \\
-x^3y-4x^2y^2-2xy^3-3y^4 \\
\underline{-x^3y-4x^2y^2-2xy^3-3y^4}
\end{array}$$

[14.] $x^2+ax+b)x^4-(a^2-b-c)x^2-(b-c)ax+bc(x^2-ax+c$

$$\begin{array}{l}
\underline{x^4+ax^3+bx^2} \\
-ax^3-(a^2-c)x^2-(b-c)ax \\
\underline{-ax^3-a^2x^2-abx} \\
\qquad cx^2+acx+bc \\
\qquad \underline{cx^2+acx+bc}
\end{array}$$

[15.]

$x^3+2ax^2+2a^2x+a^3)x^6-a^6(x^3-2ax^2+2a^2x-a^3$

$$\begin{array}{l}
\underline{x^6+2ax^5+2a^2x^4+a^3x^3} \\
-2ax^5-2a^2x^4-a^3x^3 \\
\underline{-2ax^5-4a^2x^4-4a^3x^3-2a^4x^2} \\
\qquad 2a^2x^4+3a^3x^3+2a^4x^2 \\
\qquad \underline{2a^2x^4+4a^3x^3+4a^4x^2+2a^5x} \\
\qquad\qquad -a^3x^3-2a^4x^2-2a^5x-a^6 \\
\qquad\qquad \underline{-a^3x^3-2a^4x^2-2a^5x-a^6}
\end{array}$$

In the preceding example, the term $a^6$, in the dividend, is kept there till the last step of the work, as it is easily foreseen that no use can be made of it before.

[16.] $a+x)a\quad(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c.

$$\begin{array}{l}
\underline{a+x} \\
-x \\
\underline{-x-\frac{x^2}{a}} \\
\qquad \frac{x^2}{a} \\
\qquad \underline{\frac{x^2}{a}+\frac{x^3}{a^2}} \\
\qquad\quad -\frac{x^3}{a^2}, \text{ \&c.}
\end{array}$$

In examples such as this, where the terms of the quotient are seen to succeed one another according to an obvious law, it is unnecessary to extend the operation. It is plain that the process here is interminable: if it be considered as extended till $n$ terms are got in the quotient, then the quotient, together with the fractional correction of it, may be written in this general form, namely:—

$$\frac{a}{a+x}=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}\;\cdots\cdots\left(-\frac{x}{a}\right)^{n-1}+\frac{(-x)^n}{a^{n-1}(a+x)}.$$

As particular instances, let $n$ be 2, then 3, then 4, &c.; the general result gives, in these particular cases, the several quotients

$$=1-\frac{x}{a}+\frac{x^2}{a(a+x)},$$

$$=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^2(a+x)},$$

$$=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^3(a+x)}.$$

Perhaps the learner may not see why, in the general form above, the last or $n$th term of the quotient is written $\left(-\frac{x}{a}\right)^{n-1}$. The reason is this: we see that the terms of the quotient are alternately *plus* and *minus;* the odd terms are plus, and the even terms minus; the $n$th term is therefore + or —, according as the exponent $n-1$ is even or odd. Now we know, from the rule of signs, that the putting *minus* before a quantity does not prevent an *even* power of it from being positive; though the *odd* powers are negative: the minus sign is therefore put before $\frac{x}{a}$ in order that the power, marked by the exponent $n-1$ may have the proper sign. For a like reason the minus is put before $x$ in the remainder.

[17.]
$$\begin{array}{l} x^3-2x^2+2x-1)\,x^3-1\,(1 \\ \qquad\qquad\qquad\qquad\quad \underline{x^3-2x^2+2x-1} \\ \qquad\qquad\qquad\qquad\qquad\quad 2x^2-2x \end{array}$$

This remainder is $2x(x-1)$; and the divisor itself is $(x^2-x+1)(x-1)$, so that the quotient of the remainder by the divisor is the same as the quotient of $2x$ by $x^2-x+1$. If we stop the operation, therefore, after the first step above, the quotient with the fractional correction will be $1+\dfrac{2x}{x^2-x+1}$: but the division may be continued as in the last example, as far as we please.

[18.]
$$\begin{array}{l} a^n-b^n)\,a^{m+n}-a^mb^n+a_nb^m-b^{m+n}\,(a^m+b^m \\ \qquad\qquad \underline{a^{m+n}-a^mb^n\qquad\qquad\qquad\qquad} \\ \qquad\qquad\qquad\qquad\quad a^nb^m-b^{m+n} \\ \qquad\qquad\qquad\qquad\quad \underline{a^nb^m-b^{m+n}} \end{array}$$

[19.]
$$\begin{array}{l} x+1)\,x^3+\frac{3}{4}x^2+\frac{3}{4}x+1\,(x^2-\frac{1}{4}x+1 \\ \qquad\quad \underline{x^3+\ \ x^2\qquad} \\ \qquad\qquad -\frac{1}{4}x^2+\frac{3}{4}x \\ \qquad\qquad \underline{-\frac{1}{4}x^2-\frac{1}{4}x\quad} \\ \qquad\qquad\qquad\qquad x+1 \\ \qquad\qquad\qquad\qquad \underline{x+1} \end{array}$$

$$\begin{array}{l} \text{Or, } x+1)\,4x^3+3x^2+3x+4\,(4x^2-x+4 \\ \qquad\qquad \underline{4x^3+4x^2\qquad\quad} \qquad\qquad \therefore\ x^2-\frac{1}{4}x+1=\text{quotient.} \\ \qquad\qquad\quad -x^2+3x \\ \qquad\qquad\quad \underline{-x^2-\ \ x\qquad} \\ \qquad\qquad\qquad\qquad 4x+4 \\ \qquad\qquad\qquad\qquad \underline{4x+4} \end{array}$$

It will often be found convenient, when there are fractional coefficients either in dividend or divisor, to multiply the terms by some number that will remove them. If we thus multiply the *dividend*, as in the second operation above, we must then, of course, *divide* the quotient by the number; but if we multiply the divisor, we must *also* multiply the quotient to bring all right, as is obvious.

[20.]
$$\begin{array}{r}
x^6+x^4+x^2)x^8-x^2(x^2-1 \\
\underline{x^8+x^6+x^4\phantom{-x^2}} \\
-x^6-x^4-x^2 \\
\underline{-x^6-x^4-x^2}
\end{array}$$

As the factor $x^2$ is common to both dividend and divisor, it may be expunged, and the work given thus:—

$$\begin{array}{r}
x^4+x^2+1)x^6-1(x^2-1 \\
\underline{x^6+x^4+x^2\phantom{-1}} \\
-x^4-x^2-1 \\
\underline{-x^4-x^2-1}
\end{array}$$

[21.]
$$\begin{array}{l}
1-5x)1+3x(1+8x+40x^2+200x^3+ \text{\&c.} \\
\phantom{1-5x)}\underline{1-5x} \\
\phantom{1-5x)1+3}8x \\
\phantom{1-5x)1+3}\underline{8x-40x^2} \\
\phantom{1-5x)1+3x-}40x^2 \\
\phantom{1-5x)1+3x-}\underline{40x^2-200x^3} \\
\phantom{1-5x)1+3x-40x^2-}200x^3
\end{array}$$

[22.] First,
$$\begin{array}{r}
x^2-ax+a^2)x^3+a^3(x+a \\
\underline{x^3-ax^2+a^2x\phantom{+a^3}} \\
ax^2-a^2x+a^3 \\
\underline{ax^2-a^2x+a^3}
\end{array}$$

Second,
$$\begin{array}{r}
x^2+ax+a^2)x^3-a^3(x-a \\
\underline{x^3+ax^2+a^2x\phantom{-a^3}} \\
-ax^2-a^2x-a^3 \\
\underline{-ax^2-a^2x-a^3}
\end{array}$$

Therefore, multiplying the two quotients together: $(x+a)(x-a)=x^2-a^2$. Or we may proceed thus: $(x^3+a^3)(x^3-a^3)=x^6-a^6$; and $(x^2+ax+a^2)(x^2-ax+a^2)=\{(x^2+a^2)+ax\}\{(x^2+a^2)-ax\}=(x^2+a^2)^2-a^2x^2$ (by Theorem 1, page 44) $=x^4+2a^2x^2+a^4-a^2x^2$ (by Theorem 2): the operation is as follows:—

$$\begin{array}{l}
x^4+a^2x^2+a^4)x^6-a^6(x^2-a^2 \\
\qquad\qquad\qquad \underline{x^6+a^2x^4+a^4x^2} \\
\qquad\qquad\qquad\quad -a^2x^4-a^4x^2-a^6 \\
\qquad\qquad\qquad\quad \underline{-a^2x^4-a^4x^2-a^6}
\end{array}$$

[23.] $$\begin{array}{l}
a^m+b^m)a^{m+n}+a^mb^n+a^nb^m+b^{m+n}(a^n+b^n \\
\qquad\qquad \underline{a^{m+n}+a^nb^m} \\
\qquad\qquad\qquad a^mb^n+b^{m+n} \\
\qquad\qquad\qquad \underline{a^mb^n+b^{m+n}}
\end{array}$$

Looking at the dividend, we see that it remains unaltered, though we change *m* into *n*, and *n* into *m*; but the product of quotient and divisor *is* the dividend always: consequently, when the divisor is $a^n+b^n$, we may *infer*, without any additional work, that the quotient will be $a^m+b^m$.

[24.] $$\begin{array}{l}
x-a)x^2+px+q(x+a+p \\
\qquad\quad \underline{x^2+ax} \\
\qquad\qquad (a+p)x+q \\
\qquad\qquad \underline{(a+p)x-a^2-pa}
\end{array}$$

Remainder $a^2+pa+q$, which is the same as the dividend, when the *x* is removed and *a* put in its place.

[25.]

$$\begin{array}{l}
x-a)x^4+px^3+qx^2+rx+s\,(x^3+(a+p)x^2+(a^2+pa+q)x+a^3+pa^2+qa+r \\
\qquad \underline{x^4-ax^3} \\
\qquad (a+p)x^3+qx^2 \\
\qquad \underline{(a+p)x^3-(a^2+pa)x^2} \\
\qquad\qquad\quad (a^2+pa+q)x^2+rx \\
\qquad\qquad\quad \underline{(a^2+pa+q)x^2-(a^3+pa^2+qa)x} \\
\qquad\qquad\qquad\qquad\qquad (a^3+pa^2+qa+r)x+s \\
\qquad\qquad\qquad\qquad\qquad \underline{(a^3+pa^2+qa+r)x-(a^4+pa^3+qa^2+ra} \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{Remainder, } \underline{a^4+pa^3+qa^2+ra+s}
\end{array}$$

This might have conveniently been worked by detached coefficients. (See foot-note, p. 22.)

This remainder is obviously the same as the dividend, when the *x* in the latter is replaced by *a*.

The interesting property proved in this and the preceding example, is of considerable use in the numerical solution of equations, as the learner will see when he reaches Chapter IX. of the book, where an expeditious method of finding the remainder is given.

[26.] In algebraical division, multiplication, &c. the *letters* employed may have any meaning we please to give them. It is plain that the *operations* are not at all affected by the *meaning* we give to the letters. In this and the former example, $a$ may be replaced by any *negative* value, as well as by any positive value. We may therefore put $-a'$ for $a$, throughout the work, and all will be equally correct: the remainder above will then be $a'^4 - pa'^3 + qa'^2 - ra' + s$, the divisor being $x - (-a')$; that is, $x + a'$: and if now the symbol $a'$, which may still be *anything*, be changed for $a$, the divisor will be $x + a$, and the remainder $a^4 - pa^3 + qa^2 - ra + s$, which is the same as the dividend, when the $x$ in it is replaced by $-a$. The learner however may, if he please, go through the actual division by $x + a$, in imitation of the process above. (See the remarks at page 56 of the book.)

---

## PAGE 57.

[1.] Putting 3 for $x$, in $x^4 - 3x^3 - 2x^2 + x - 6$, it becomes $3^4 - 3.3^3 - 2.3^2 + 3 - 6 = 81 - 81 - 18 + 3 - 6 = -21$, the remainder sought. The *general* operation above *proves* this; but it may be satisfactory to the learner to verify the particular case here proposed by actual division, as follows:—

$$
\begin{array}{rl}
x - 3\,) & x^4 - 3x^3 - 2x^2 + x - 6\ (\,x^3 - 2x - 5 \\
 & \underline{x^4 - 3x^3} \\
 & \qquad\qquad -2x^2 + x \\
 & \qquad\qquad \underline{-2x^2 + 6x} \\
 & \qquad\qquad\qquad\quad -5x - 6 \\
 & \qquad\qquad\qquad\quad \underline{-5x + 15} \\
\text{Remainder,} & \qquad\qquad\qquad\qquad \underline{-21}
\end{array}
$$

[2.] Putting $-1$ for $x$, in $3x^3 + 6x^2 - 8x + 1$, it becomes $-3 + 6 + 8 + 1 = 12$, the remainder sought. The learner may verify this by actual division as above.

[3.] For $x=-4$, $6x^4+2x^3-4x^2-3x-4$, becomes $6.4^4-2.4^3-4.4^2+3.4-4=1352$, the remainder sought. Let us verify this by division:—

$$
\begin{array}{rlllll}
x+4\,) & 6x^4+2x^3 & -4x^2 & -3x & -4 & (\,6x^3-22x^2+84x-339 \\
 & \underline{6x^4+24x^3} & & & & \\
 & \quad -22x^3 & -4x^2 & & & \\
 & \quad \underline{-22x^3} & \underline{-88x^2} & & & \\
 & & 84x^2 & -3x & & \\
 & & \underline{84x^2} & \underline{+336x} & & \\
 & & & -339x & -4 & \\
 & & & \underline{-339x} & \underline{-1356} & \\
 & & \text{Remainder,} & & 1352 &
\end{array}
$$

Detached coefficients might have been used in this and in the preceding example.

[4.] For $x=2$, $5x^6-2x^4+3x^2-1$, becomes $5.2^6-2.2^4+3.2^2-1=5.64-2.16+3.4-1=299$, the remainder sought.

[5.] For $x=-3$, $x^5+3^5$, is $-3^5+3^5$ or $0$: hence there is *no* remainder.

[6.] For $x=3$, $x^5-3^5$, is $3^5-3^5$ or $0$: hence there is *no* remainder.

[7.], [8.] In these examples it is plain that whether *plus* 2 or *minus* 2 be put for $x$, the expression becomes 0, so that there is no remainder.

[9.], [10.] In these examples, the substitution of either *plus* or *minus* 2 gives the same result, namely $2^6+2^6$, that is, $2^6(1+1)=2^7$, which is therefore the remainder.

---

## INVOLUTION.—PAGE 62.

[1.] $(x-3y)^2=x^2-6xy+9y^2$; by Theorem 2, page 44.*

$$
\begin{array}{rl}
 & \underline{x\ -3y} \\
 & x^3-6x^2y+9xy^2 \\
 & \underline{\qquad\ -3x^2y+18xy^2-27y^3} \\
(x-3y)^3= & \underline{x^3-9x^2y+27xy^2-27y^3}
\end{array}
$$

---

* Theorems 1 and 2, at page 44 of the Algebra, will be very frequently employed in these examples.

[2.] $(2a+3)^2 = 4a^2+12a+9$

$$\begin{array}{r}
4a^2+12a+9 \\
2a+3 \\
\hline
8a^3+24a^2+18a \phantom{{}+27} \\
12a^2+36a+27 \\
\hline
(2a+3)^3 = 8a^3+36a^2+54a+27 \\
\hline
\end{array}$$

[3.] $[(a+2b)-3c]^2 = (a+2b)^2-6c(a+2b)+9c^2 = a^2+4ab+4b^2-6ac-12bc+9c^2.$

[4.] $[(x^2+2x)-1]^2 = (x^2+2x)^2-2(x^2+2x)+1 = x^4+4x^3+4x^2-2x^2-4x+1 = x^4+4x^3+2x^2-4x+1.$

[5.] $[(1+x)(1-x)]^3 = (1-x^2)^3;\ (1-x^2)^2 = 1-2x^2+x^4$

$$\begin{array}{r}
1-2x^2+x^4 \phantom{{}-x^6} \\
1-x^2 \phantom{{}+x^4-x^6} \\
\hline
1-2x^2+x^4 \phantom{{}-x^6} \\
-x^2+2x^4-x^6 \\
\hline
(1-x^2)^3 = 1-3x^2+3a^4-x^6 \\
\hline
\end{array}$$

[6.] $\{(x-y)^2\}^3 = (x^2-2xy+y^2)^3;\ \{(x^2-2xy)+y^2\}^2 =$

$(x^2-2xy)^2+2(x^2-2xy)y^2+y^4 =$

$x^4-4x^3y+4x^2y^2+2x^2y^2-4xy^3+y^4 =$

$$\begin{array}{r}
x^4-4x^3y+6x^2y^2-4xy^3-y^4 \phantom{{}-2xy^5+y^6} \\
x^2-2xy+y^2 \phantom{{}+6x^2y^2-4xy^3-y^4-2xy^5+y^6} \\
\hline
x^6-4x^5y+6x^4y^2-4x^3y^3+x^2y^4 \phantom{{}-2xy^5+y^6} \\
-2x^5y+8x^4y^2-12x^3y^3+8x^2y^4-2xy^5 \phantom{{}+y^6} \\
x^4y^2-4x^3y^3+6x^2y^4-4xy^5+y^6 \\
\hline
(x-y)^6 = x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6 \\
\hline
\end{array}$$

[7.] $\{(x^2-2ax)+3a^2\}^2 = (x^2-2ax)^2+6a^2(x^2-2ax)+9a^4$

$= x^4-4ax^3+4a^2x^2+6a^2x^2-12a^3x+9a^4$

$= x^4-4ax^3+10a^2x^2-12a^3x+9a^4.$

[8.]

$[(a+b)+(c+d)]^2 = (a+b)^2+2(a+b)(c+d)+(c+d)^2$

$= a^2+2ab+b^2+2(ac+bc+ad+bd)+c^2+2cd+d^2$

$= a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2.$

[9.]

$$\begin{aligned}(1-2x+x^2+x^2+x^4)^2 &= \{(1-2x)+(2x^2+x^4)\}^2 \\ &= (1-2x)^2+2(1-2x)(2x^2+x^4)+(2x^2+x^4)^2 \\ &= 1-4x+4x^2+2x^2(1-2x)(2+x^2)+4x^4+4x^6+x^8 \\ &= 1-4x+4x^2+2x^2(2-4x+x^2-2x^3)+4x^4+4x^6+x^8 \\ &= 1-4x+8x^2-8x^3+6x^4-4x^5+4x^6+x^8.\end{aligned}$$

[10.] The expression enclosed between the outer brackets is,

$$(1-x)^2(1+x)^2=(1-x^2)^2=1-2x^2+x^4;$$

the square of this is,

$$\{(1-2x^2)+x^4\}^2=(1-2x^2)^2+2(1-2x^2)x^4+x^8=$$
$$1-4x^2+4x^4+2x^4-4x^6+x^8=1-4x^2+6x^4-4x^6+x^8;$$

and the square of this square is, of course, the required fourth power: we shall compute it by using only detached coefficients, as explained at page 15 of this Key; the learner observing that the terms of each factor follow one another according to the powers of $x^2$.

$$\begin{array}{rrrrrrrrr}
1 & -4 & +6 & -4 & +1 & & & & \\
1 & -4 & +6 & -4 & +1 & & & & \\
\hline
1 & -4 & +6 & -4 & +1 & & & & \\
 & -4 & +16 & -24 & +16 & -4 & & & \\
 & & 6 & -24 & +36 & -24 & +6 & & \\
 & & & -4 & +16 & -24 & +16 & -4 & \\
 & & & & 1 & -4 & +6 & -4 & +1 \\
\hline
1 & -8 & +28 & -56 & +70 & -56 & +28 & -8 & +1. \\
\hline
\end{array}$$

Consequently, by now introducing $x^2$ and its powers, we find the result sought to be

$$1-8x^2+28x^4-56x^6+70x^8-56x^{10}+28x^{12}-8x^{14}+x^{16}.$$

The learner must clearly see the advantage of using detached coefficients in a long multiplication like this: besides the time and space saved by the suppression of the letters, all that confusion is avoided that would otherwise be presented to the eye by so many exponents.

[11.] The expression within the outer brackets is $(x+a)^2$ $(x-a)^2=x)^2-a^2)^2=x^4-{}^2a^2x2+a4$, so that we have to take the cube of this expression, which may be replaced by $x^4(1-2\frac{a^2}{x^2}+\frac{a^4}{x^4})$, where the quantity within the brackets has its terms proceeding according to the powers of $\frac{a^2}{x^2}$: we may therefore use the coefficients only, as in last example.

$$\begin{array}{rrrrrrr}
1-2 & +1 & & & & & \\
1-2 & +1 & & & & & \\
\hline
1-2 & +1 & & & & & \\
-2 & +4 & -2 & & & & \\
 & 1 & -2 & +1 & & & \\
\hline
1-4 & +6 & -4 & +1 & & & \\
1-2 & +1 & & & & & \\
\hline
1-4 & +6 & -4 & +1 & & & \\
-2 & +8 & -12 & +8 & -2 & & \\
 & 1 & -4 & +6 & -4 & +1 & \\
\hline
1-6 & +15 & -20 & +15 & -6 & +1. & \\
\hline
\end{array}$$

These are the coefficients of $\frac{a^2}{x^2}$ and its powers, commencing with the second coefficient $-6$; and when these powers are introduced the whole must be multiplied by $x^{12}$, that is, by the third power of $x^4$, the factor that has been omitted: it is plain therefore that the complete result will be

$$x^{12}-6a^2x^{10}+15a^4x^8-20a^6x^6+15a^8x^4-6a^{10}x^2+a^{12}.$$

[12.] $(a-b)^3=a^3-3a^2b+3ab^2-b^3$ (page 61, Algebra); therefore the proposed expression is $(a^3+2ab^2-b^3)^2$

$$\begin{array}{llllll}
a^3 & +2ab^2 & -b^3 & & & \\
a^3 & +2ab^2 & -b^3 & & & \\
\hline
a^6 & +2a^4b^2 & -a^3b^3 & & & \\
 & 2a^4b^2 & & +4a^2b^4 & -2ab^5 & \\
 & & -a^3b^3 & & -2ab^5 & +b^6 \\
\hline
a^6 & +4a^4b^2 & -2a^3b^3 & +4a^2b^4 & -4ab^5 & +b^6. \\
\hline
\end{array}$$

Or, $a^6\left(1+0\frac{b}{a}+2\frac{b^2}{a^2}-\frac{b^3}{a^3}\right)^2$, in which form we may use detached coefficients, as in last example.

$$\begin{array}{rrrrrrr}
1 & +0 & +2 & -1 & & & \\
1 & +0 & +2 & -1 & & & \\
\cline{1-4}
1 & +0 & +2 & -1 & & & \\
 & & 2 & +0 & +4 & -2 & \\
 & & & -1 & -0 & -2 & +1 \\
\hline
1 & +0 & +4 & -2 & +4 & -4 & +1.
\end{array}$$

These are the coefficients of $\frac{b}{a}$ and its powers, commencing with the second coefficient 0; therefore, when the factor $a^6$ is introduced, we shall have the result obtained in full by the common method.

[13.] By p. 61 (Algebra) $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

The sum of these is, $2a^3+6ab^2$, of which the square is $4a^6+24a^4b^2+36a^2b^4$, by Theorem 2, p. 44 (Algebra).

[14.] $2x-(3x-2)=2-x$ and $(2-x)^5=(2-x)^2(2-x)^2(2-x)=(4-4x+x^2)^2(2-x)$. By detached coefficients,

$$\begin{array}{rrrrrr}
4 & -4 & +1 & & & \\
4 & -4 & +1 & & & \\
\cline{1-3}
16 & -16 & +4 & & & \\
 & -16 & +16 & -4 & & \\
 & & 4 & -4 & +1 & \\
\cline{1-5}
16 & -32 & +24 & -8 & +1 & \\
2 & -1 & & & & \\
\cline{1-5}
32 & -64 & +48 & -16 & +2 & \\
 & -16 & +32 & -24 & +8 & -1 \\
\hline
32 & -80 & +80 & -40 & +10 & -1.
\end{array}$$

And, introducing the powers of $x$, the complete result is,

$$32-80x+80x^2-40x^3+10x^4-x^5.$$

[15.] $(a+b+c)^2 = (a+b)^2 + 2(a+b)c + c^2$; and subtracting $(a+b)^2 + c^2$, there remains $2(a+b)c$, of which the cube is $8(a^3 + 3a^2b + 3ab^2 + b^3)c^3$.

[16.] $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$; and subtracting $3x^2y + 3xy^2$, there remains $x^3 + y^3$, the square of which is $x^6 + 2x^3y^3 + y^6$.

[17] $(x^2+a^2)^2 = x^4 + 2a^2x^2 + a^4$; and subtracting $4a^2x^2$, there remains $x^4 - 2a^2x^2 + a^4$; the square of which (see example 11) is $x^8 - 4a^2x^6 + 6a^4x^4 - 4a^6x^2 + a^8$.

[18.] $(3x-2)^2 = 9x^2 - 12x + 4$; and $(2x-3)^2 = 4x^2 - 12x + 9$: subtracting this there remains $5x^2 - 5 = 5(x^2-1)$, the cube of which is $125(x^6 - 3x^4 + 3x^2 - 1)$.

[19.] $(5x+1)^2 = 25x^2 + 10x + 1$; and $(3x+1)^2 = 9x^2 + 6x + 1$: subtracting this there remains $16x^2 + 4x = 4x(4x+1)$, the cube of which is $64x^3(64x^3 + 48x^2 + 12x + 1)$.

[20.] $(4-x)^2 = 16 - 8x + x^2$; and $(3+2x)^2 = 9 + 12x + 4x^2$: subtracting this there remains $7 - 20x - 3x^2$, which is to be cubed. We shall use detached coefficients.

$$\begin{array}{rrrrrrr}
7- & 20- & 3 & & & & \\
7- & 20- & 3 & & & & \\
\hline
49- & 140- & 21 & & & & \\
- & 140+ & 400+ & 60 & & & \\
 & - & 21+ & 60+ & 9 & & \\
\hline
49- & 280+ & 358+ & 120+ & 9 & & \\
7- & 20- & 3 & & & & \\
\hline
343- & 1960+ & 2506+ & 840+ & 63 & & \\
- & 980+ & 5600- & 7160- & 2400- & 180 & \\
 & - & 147+ & 840- & 1074- & 360- & 27 \\
\hline
343- & 2940+ & 7959- & 5480- & 3411- & 540- & 27. \\
\hline
\end{array}$$

Therefore, introducing the powers of $x$, commencing with the second coefficient, the required cube is

$$343 - 2940x + 7959x^2 - 5480x^3 - 3411x^4 - 540x^5 - 27x^6.$$

---

Although in many of the more troublesome examples in the present Section we have, for the sake of abbreviation, employed the method of detached coefficients, yet we have not thought it prudent to advert to this method in the *Algebra*. Our reason is this: the principal object with a teacher, in giving practical examples in the early part of Algebra, is to habituate the student to the use of *symbols*, and to give him a facility in working with them; an object that would not be subserved by suppressing the *letters*, and working only with *figures*. But in the present Key, we have thought it instructive to give the foregoing specimens of the shorter mode of proceeding, which might have been adopted more frequently, if brevity of work were the only thing necessary to secure.

Page 68. The examples in this page require no explanation here: the answers at the end of the Algebra exhibit the solutions.

## EVOLUTION: EXTRACTION OF THE SQUARE-ROOT.

### PAGE 73.

[1.][*]

$$\begin{array}{rl}
 & 9x^4-6ax^2+a^2\,[3x^2-a \\
 & 9x^4 \\ \hline
6x^2-a] & -6ax^2+a^2 \\
 & -6ax^2+a^2 \\ \hline
\end{array}$$

[2.]

$$\begin{array}{rl}
 & x^4+2x^3+3x^2+2x+1\,[x^2+x+1 \\
 & x^4 \\ \hline
2x^2+x] & 2x^3+3x^2 \\
 & 2x^3+x^2 \\ \hline
2x^2+2x+1] & 2x^2+2x+1 \\
 & 2x^2+2x+1 \\ \hline
\end{array}$$

[3.]

$$\begin{array}{rl}
 & 9x^4-12x^3+16x^2-8x+4\,[3x^2-2x+2 \\
 & 9x^4 \\ \hline
6x^2-2x] & -12x^3+16x^2 \\
 & -12x^3+4x^2 \\ \hline
6x^2-4x+2] & 12x^2-8x+4 \\
 & 12x^2-8x+4 \\ \hline
\end{array}$$

---

[*] The learner may easily work the first thirteen of these examples by the method of detached coefficients.

$$\begin{array}{rl}
[4.] & 16a^2x^2 + 8ax^2 + x^2 + 8ax + 2x + 1\,[4ax + x + 1 \\
 & 16a^2x^2 \\
8ax + x] & \quad 8ax^2 + x^2 \\
 & \quad 8ax^2 + x^2 \\
8ax + 2x + 1] & \qquad 8ax + 2x + 1 \\
 & \qquad 8ax + 2x + 1
\end{array}$$

$$\begin{array}{rl}
[5.] & 4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1\,[2x^3 + 3x^2 - x + 1 \\
 & 4x^6 \\
4x^3 + 3x^2] & 12x^5 + 5x^4 \\
 & 12x^5 + 9x^4 \\
4x^3 + 6x^2 - x] & \quad -4x^4 - 2x^3 + 7x^2 \\
 & \quad -4x^4 - 6x^3 + x^2 \\
4x^3 + 6x^2 - 2x + 1] & \qquad 4x^3 + 6x^2 - 2x + 1 \\
 & \qquad 4x^3 + 6x^2 - 2x + 1
\end{array}$$

$$\begin{array}{rl}
[6.] & 4x^6 - 24x^5 + 60x^4 - 80x^3 + 60x^2 - 24x + 4\,[2x^3 - 6x^2 + 6x - 2 \\
 & 4x^6 \\
4x^3 - 6x^2] & -24x^5 + 60x^4 \\
 & -24x^5 + 36x^4 \\
4x^3 - 12x^2 + 6x] & \quad 24x^4 - 80x^3 + 60x^2 \\
 & \quad 24x^4 - 72x^3 + 36x^2 \\
4x^3 - 12x^2 + 12x - 2] & \qquad -8x^3 + 24x^2 - 24x + 4 \\
 & \qquad -8x^3 + 24x^2 - 24x + 4
\end{array}$$

$$\begin{array}{rl}
[7.] & 4x^4 - 12x^3 + 11x^2 - 3x + \frac{1}{4}\,[2x^2 - 3x + \frac{1}{2} \\
 & 4x^4 \\
4x^2 - 3x] & -12x^3 + 11x^2 \\
 & -12x^3 + 9x^2 \\
4x^2 - 6x + \frac{1}{2}] & \qquad 2x^2 - 3x + \frac{1}{4} \\
 & \qquad 2x^2 - 3x + \frac{1}{4}
\end{array}$$

$$\begin{array}{rl}
[8.] & 49x^4 + 28x^3 - 17x^2 - 6x + \frac{9}{4}\,[7x^2 + 2x - \frac{3}{2} \\
 & 49x^4 \\
14x^2 + 2x] & 28x^3 - 17x^2 \\
 & 28x^3 + 4x^2 \\
14x^2 + 4x - \frac{3}{2}] & \quad -21x^2 - 6x + \frac{9}{4} \\
 & \quad -21x^2 - 6x + \frac{9}{4}
\end{array}$$

[9.]

$$\begin{array}{ll}
 & 4x^4+6x^3+\frac{89}{4}x^2+15x+25\ [2x^2+\frac{3}{2}x+5 \\
 & 4x^4 \\
4x^2+\frac{3}{2}x] & 6x^3+\frac{89}{4}x^2 \\
 & 6x^3+\frac{9}{4}x^2 \\
4x^2+3x+5] & 20x^2+15x+25 \\
 & 20x^2+15x+25
\end{array}$$

[10.]

$$\begin{array}{ll}
 & 1+4x-2x^2-4x^3+25x^4-24x^5+16x^6\ [1+2x-3x^2+4x^3 \\
 & 1 \\
2+2x] & 4x-2x^2 \\
 & 4x+4x^2 \\
2+4x-3x^2] & -6x^2-4x^3+25x^4 \\
 & -6x^2-12x^3+9x^4 \\
2+4x-6x^2+4x^3] & 8x^3+16x^4-24x^5+16x^6 \\
 & 8x^3+16x^4-24x^5+16x^6
\end{array}$$

[11.]

$$\begin{array}{ll}
 & 64x^6+192x^5+240x^4+160x^3+60x^2+12x+1\ [8x^3+12x^2+6x+1 \\
 & 64x^6 \\
16x^3+12x^2] & 192x^5+240x^4 \\
 & 192x^5+144x^4 \\
16x^3+24x^2+6x] & 96x^4+160x^3+60x^2 \\
 & 96x^4+144x^3+36x^2 \\
16x^3+24x^2+12x+1] & 16x^3+24x^2+12x+1 \\
 & 16x^3+24x^2+12x+1
\end{array}$$

[12.]

$$\begin{array}{ll}
 & 4x^8-16x^6-16x^5+12x^4+32x^3+24x^2+8x+1\ [2x^4-4x^2-4x-1 \\
 & 4x^8 \\
4x^4-4x^2] & -16x^6-16x^5 \\
 & -16x^6+16x^4 \\
4x^4-8x^2-4x] & -16x^5-4x^4+32x^3 \\
 & -16x^5+32x^3+16x^2 \\
4x^4-8x^2-8x-1] & -4x^4+8x^2+8x+1 \\
 & -4x^4+8x^2+8x+1
\end{array}$$

[13.] $x^4-7\cdot2x^3+13\cdot44x^2-1\cdot728x+\cdot0576$ [$x^2-3\cdot6x+\cdot24$

$x^4$

$2x^2-3\cdot6x$] $-7\cdot2x^3+13\cdot44x^2$

$-7\cdot2x^3+12\cdot96x^2$

$2x^2-7\cdot2x+\cdot24$] $\cdot48x^2-1\cdot728x+\cdot0576$

$\cdot48x^2-1\cdot728x+\cdot0576$

[14.]

$9a^2+6ab+30ac+6ad+b^2+10bc+2bd+25c^2+10cd+d^2$; or,

$(3a+b)^2+30ac+6ad+10bc+2bd+25c^2+10cd+d^2$ [$(3a+b)+5c+d$

$(3a+b)^2$

$6a+2b+5c$] $30ac+6ad+10bc$

$30ac+10bc+25c^2$

$6a+2b+10c+d$] $6ad+2bd+10cd+d^2$

$6ad+2bd+10cd+d^2$

[15.] $4x^2y^4-12x^3y^3+17x^4y^2-12x^5y+4x^6$ [$2xy^2-3x^2y+2x^3$

$4x^2y^4$

$4xy^2-3x^2y$] $-12x^3y^3+17x^4y^2$

$-12x^3y^3+9x^4y^2$

$4xy^2-6x^2y+2x^3$] $8x^4y^2-12x^5y+4x^6$

$8x^4y^2-12x^5y+4x^6$

[16.] $a^{4m}-4a^{3m+n}+6a^{2m+2n}-4a^{m+3n}+a^{4n}$ [$a^{2m}-2a^{m+n}+a^{2n}$

$a^{4m}$

$2a^{2m}-2a^{m+n}$] $-4a^{3m+n}+6a^{2m+2n}$

$-4a^{3m+n}+4a^{2m+2n}$

$2a^{2m}-4a^{m+n}+a^{2n}$] $2a^{2m+2n}-4a^{m+3n}+a^{4n}$

$2a^{2m+2n}-4a^{m+3n}+a^{4n}$

[17.] $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ [$x^2-2xy+y^2=(x-y)^2$ and the square root of this is $x-y$.

$x^4$

$2x^2-2xy$] $-4x^3y+6x^2y^2$

$-4x^3y+4x^2y^2$

$2x^2-4xy+y^2$] $2x^2y^2-4xy^3+y^4$

$2x^2y^2-4xy^3+y^4$

[18.]
$$\begin{array}{rl}
 & 16x^4-96x^3y+216x^2y^2-216xy^3+81y^4\,[4x^2-12xy+9y^2, \\
 & 16x^4 \\
8x^2-12xy] & -96x^3y+216x^2y^2 \\
 & -96x^3y+144x^2y^2 \\
8x^2-24xy+9y^2] & \qquad 72x^2y^2-216xy^3+81y^4 \\
 & \qquad 72x^2y^2-216xy^3+81y^4
\end{array}$$

and the square root of this is $2x-3y$.

[19.]
$$\begin{array}{rl}
 & x^4+4x^3-3x^2-8x+4\,[x^2+2x-2, \\
 & x^4 \\
2x^2+2x] & \quad 4x^3-3x^2 \\
 & \quad 4x^3+4x^2 \\
2x^2+4x-2] & \quad -7x^2-8x+4 \\
 & \quad -4x^2-8x+4 \\
 & \quad -3x^2
\end{array}$$

*Incomplete root;* consequently (pp. 71, 72, Alg.) the proposed expression is made up of the two factors below;

$$\{x^2+(2+\surd 3)x-2\}\{x^2+(2-\surd 3)x-2\}$$

[20.]
$$\begin{array}{rl}
 & x^4-4x^3+7x^2-4x+1\,[x^2-2x+1, \\
 & x^4 \\
2x^2-2x] & -4x^3+7x^2 \\
 & -4x^3+4x^2 \\
2x^2-4x+1] & \quad 3x^2-4x+1 \\
 & \quad 2x^2-4x+1 \\
 & \quad x^2
\end{array}$$

*Incomplete root;* therefore the proposed expression is formed from the two factors below;

$$\{x^2-(2-\surd-1)x+1\}\{x^2-(2+\surd-1)x+1\}$$

NOTE. At page 77 of the Algebra, a method of extracting the square root of a polynomial is proposed which is much more compact than the mode hitherto adopted, and which has uniformly been employed in the foregoing examples: we shall here apply the method alluded to, to Example 12 above, giving to the work an arrangement a little different from that in the book, but in perfect accordance with the principle there explained,—

$$\begin{array}{r}
2x^4\pm\surd(16x^6+16x^5-12x^4-32a^3-24x^2-8x-1), \\
2x^4-4x^2\pm\surd(16x^5+4x^4-32x^3-24x^2-8x-1), \\
2x^4-4x^2-4x\pm\surd(4x^4-8x^2-8x-1), \\
2x^4-4x^2-4x-1\pm\surd 0
\end{array}$$

$\therefore 2x^4-4x^2-4x-1$ is the root.

Again, let Example 20 be taken: we proceed thus:—

$$\begin{array}{r}
x^2 \pm \sqrt{}(4x^3 - 7x^2 + 4x - 1), \\
x^2 - 2x \pm \sqrt{}(-3x^2 + 4x - 1), \\
x^2 - 2x + 1 \pm \sqrt{}(-x^2). \therefore \{x^2 - (2 - \sqrt{-1})x + 1\}\{x^2 - (2 + \sqrt{-1})x + 1\}
\end{array}$$

are the two factors of which the proposed expression is formed.

The *signs* of the terms under the radical are changed, in the above process, in order that, at each step, the two factors which reproduce the expression proposed may be exhibited with the proper signs. In the former mode of working this example, the final remainder, $x^2$, must have its sign changed when introduced, under the radical, in the incomplete root. The learner will easily see that, whichever *pair* of expressions above be multiplied together, the original expression will be produced; remembering that the sum, multiplied by the difference of two quantities, gives the difference of their squares. Thus,—

$$\{x^2 - 2x + \sqrt{}(-3x^2 + 4x - 1)\}\{x^2 - 2x - \sqrt{}(-3x^2 + 4x - 1)\}$$

gives the same product as the factors above.

---

## EXTRACTION OF THE CUBE ROOT.—PAGE 76.

[1.]

$$\begin{array}{r|l}
 & x^3 - 9x^2 + 27x - 27\,[x - 3 \\
 & x^3 \\
\hline
3x^2 - 9x + 9\,] & -9x^2 + 27x - 27 \\
 & \underline{-9x^2 + 27x - 27}
\end{array}$$

[2:]

$$\begin{array}{r|l}
 & x^6 - 3x^5 + 5x^3 - 3x - 1\,[x^2 - x - 1 \\
 & x^6 \\
\hline
3x^4 - 3x^3 + x^2\,] & -3x^5 + 0x^4 + 5x^3 \\
 & -3x^5 + 3x^4 - \phantom{0}x^3 \\
\hline
3(x^2 - x)^2 - 3(x^2 - x) + 1\,] & -3x^4 + 6x^3 + 0x^2 - 3x - 1 \\
\text{or, } 3x^4 - 6x^3 + 0x^2 + 3x + 1\,] & \underline{-3x^4 + 6x^3 + 0x^2 - 3x - 1}
\end{array}$$

$$\begin{array}{rl}
[3.] & x^6+6x^5-40x^3+96x-64\,[x^2+2x-4\\
 & x^6\\ \hline
3x^4+6x^3+4x^2\,] & 6x^5+0x^4-40x^3\\
 & 6x^5+12x^4+8x^3\\ \hline
3(x^2+2x)^2-12(x^2+2x)+16\,] & -12x^4-48x^3+0x^2+96x-64\\
\text{or, } 3x^4+12x^3+0x^2-24x+16\,] & -12x^4-48x^3+0x^2+96x-64\\ \hline
\end{array}$$

$$\begin{array}{rl}
[4.] & x^6-6x^5+15x^4-20x^3+15x^2-6x+1\,[x^2-2x+1\\
 & x^6\\ \hline
3x^4-6x^3+4x^2\,] & -6x^5+15x^4-20x^3\\
 & -6x^5+12x^4-8x^3\\ \hline
3(x^2-2x)^2+3(x^2-2x)+1\,] & 3x^4-12x^3+15x^2-6x+1\\
\text{or, } 3x^4-12x^3+15x^2-6x+1\,] & 3x^4-12x^3+15x^2-6x+1\\ \hline
\end{array}$$

$$\begin{array}{rl}
[5.] & x^6-12x^5+60x^4-160x^3+240x^2-192x+64\\
 & x^6 \qquad\qquad\qquad\qquad\qquad\qquad [x^2-4x+4\\ \hline
3x^4-12x^3+16x\,] & -12x^5+60x^4-160x^3\\
 & -12x^5+48x^4-64x^3\\ \hline
3(x^2-4x)^2+12(x^2-4x)+16\,] & 12x^4-96x^3+240x^2-192x+64\\
\text{or, } 3x^4-24x^3+60x^2-48x+16\,] & 12x^4-96x^3+240x^2-192x+64\\ \hline
\end{array}$$

$$\begin{array}{rl}
[6.] & x^6-3ax^5+6a^2x^4-7a^3x^3+6a^4x^2-3a^5x+a^6\\
 & x^6 \qquad\qquad\qquad\qquad\qquad\qquad [x^2-ax+a^2\\ \hline
3x^4-3ax^3+a^2x^2\,] & -3ax^5+6a^2x^4-7a^3x^3\\
 & -3ax^5+3a^2x^4-a^3x^2\\ \hline
3(x^2-ax)^2+3a^2(x^2-ax)+a^4\,] & 3a^2x^4-6a^3x^2+6a^4x^2-3a^5x+a^6\\
\text{or, } 3x^4-6ax^3+6a^2x^2-3a^3x+a^4\,] & 3a^2x^4-6a^3x^2+6a^4x^2-3a^5x+a^6\\ \hline
\end{array}$$

[7.]

$$\begin{array}{rl}
 & 8x^6+48x^5y+60x^4y^2-80x^3y^3-90x^2y^4+108xy^5-27y^6\,[2x^2+\\
 & 8x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4xy-3y^2\\ \hline
12x^4+24x^3y+16x^2y^2\,] & 48x^5y+60x^4y^2-80x^3y^3\\
 & 48x^5y+96x^4y^2+64x^3y^3\\ \hline
3(2x^2+4xy)^2-18x^2y^2-36xy^3+9y^4\,] & -36x^4y^2-144x^3y^3-90x^2y^4+108xy^5-27y^6\\
\text{or, } 12x^4+48x^3y+30x^2y^2-36xy^3+9y^4\,] & -36x^4y^2-144x^3y^3-90x^2y^4+108xy^5-27y^6\\ \hline
\end{array}$$

$$\begin{array}{rl}
[8.] & a^{3m}-6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n}\,[a^m-\\
 & a^{3m} \qquad\qquad\qquad\qquad\qquad\qquad\qquad 2ax^n\\ \hline
3a^{2m}-6a^{m+1}x^n+4a^2x^{2n}\,] & -6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n}\\
 & -6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n}\\ \hline
\end{array}$$

[10.]

$$\begin{array}{r|l}
 & x^9+3x^8+6x^7+10x^6+12x^5+12x^4+10x^3+6x^2+3x+1 \quad [x^3+x^2+x+1 \\
 & x^9 \\
\hline
3x^6+3x^5+x^4 & 3x^8+6x^7+10x^6 \\
 & 3x^8+3x^7+x^6 \\
\hline
3(x^3+x^2)^2+3x^4+3x^3+x^2 & 3x^7+9x^6+12x^5+12x^4+10x^3 \\
\text{or, } 3x^6+6x^5+6x^4+3x^3+x^2 & 3x^7+6x^6+6x^5+3x^4+x^3 \\
\hline
3(x^3+x^2+x)^2+3x^3+3x^2+3x+1 & 3x^6+6x^5+9x^4+9x^3+6x^2+3x+1 \\
\text{or, } 3x^6+6x^5+9x^4+9x^3+6x^2+3x+1 & 3x^6+6x^5+9x^4+9x^3+6x^2+3x+1 \\
\hline
\end{array}$$

## REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR.—PAGE 86.

[1.] In this example the changed numerators are $15(2x+3)$, $12x$, $20$; and the common denominator, $60$; therefore the changed fractions are $\frac{30x+45}{60}$, $\frac{12x}{60}$, $\frac{20}{60}$.

[2.] The new numerators are, $3ac$, $2a^2bc$, $6a^3b$; and the common denominator, $6bc$: therefore the changed fractions are $\frac{3ac}{6bc}$, $\frac{2a^2bc}{6bc}$, $\frac{6a^3b}{6bc}$.

[3.] The denominators of the first two fractions are made alike, by multiplying the first by 2 and the second by $b$: therefore we have only to multiply the denominator of the third, that is of $\frac{3}{1}$ by $6b$, to render all the denominators alike: hence, multiplying numerator and denominator of each by the same thing, we have for the changed fractions,

$$\frac{2ax}{6b},\ \frac{6^2x^2}{6b},\ \frac{18b}{6b}.$$

[4.] The denominators of the first and second fractions are made alike by multiplying the former by $b$ and the latter by $a$: hence to make all alike, we must multiply each of these again by 4 and the third by $abx$: the changed fractions are thus $\frac{16b}{4abx}$, $\frac{8a^2}{4abx}$, $\frac{3abx}{4abx}$.

[5.] The first two denominators multiplied together make the third (theorem 1, page 44, Algebra): therefore we have only to multiply the terms of the first fraction by $a+x$, and the terms of the second by $a-x$, and to leave the third untouched: the changed fractions are thus,

$$\frac{(a+x)^2}{a^2-x^2},\quad \frac{(a-x)^2}{a^2-x^2},\quad \frac{1}{a^2-x^2}.$$

[6.] The common denominator here is evidently $xyz$, so that the terms of the first fraction are to be multiplied by $z$, those of the second by $y$, and those of the third by $x$: the fractions are, therefore, $\frac{z}{xyz}, \frac{2y}{xyz}, \frac{3x}{xyz}$.

[7.] These fractions are, $\frac{x}{(1+x)(1-x)}, \frac{1}{(1-x)(1-x)}$: therefore the denominators are made alike, by multiplying the first by $1-x$, and the second by $1+x$; so that the changed fractions are, $\frac{x(1-x)}{(1+x)(1-x)^2}, \frac{1+x}{(1+x)(1-x)^2}$.

[8.] These fractions are the same as

$$\frac{x}{a(a+x)},\ -\frac{a}{x(a-x)},\ \frac{1}{a^2-x^2};$$

and it is plain that by multiplying the first denominator by $a-x$, and the second by $a+x$, the factors within the brackets will each be the same as the third denominator. Consequently if, after these multiplications, we multiply the first denominator by $x$, the second by $a$, and the third by $ax$, *all* the denominators will be made alike; hence the terms of the first fraction must be multiplied by $x(a-x)$, those of the second by $a(a+x)$, and those of the third by $ax$: the changed fractions thus become

$$\frac{x^2(a-x)}{ax(a^2-x^2)},\ -\frac{a^2(a+x)}{ax(a^2-x^2)},\ \frac{ax}{ax(a^2-x^2)}.$$

The *minus*, before the middle fraction, might of course have been spared by changing the sign of the denominator, writing the fraction thus, $\frac{a^2(a+x)}{ax(x^2-a^2)}$.

[9.] The denominators of the third and fourth fractions are $(a+x)(a+x)$ and $6(a+x)(a-x)$, which are made alike by multiplying the first of these by $6(a-x)$, and the second by $a+x$; for each then becomes $6(a+x)^2(a-x)$: hence to make *all* alike, it is plain that we shall only have to multiply the first denominator $2(a+x)$, by $3(a+x)(a-x)$, and the second $3(a-x)$, by $2(a+x)^2$. Consequently, by multiplying the terms of the first fraction by $3(a^2-x^2)$, those of the second by $2(a+x)^2$, those of the third by $6(a-x)$, and those of the fourth by $a+x$, we shall have the following equivalent fractions in a common denominator:

$$\frac{3(a^2-x^2)}{6(a-x)(a+x)^2},\ \frac{4(a+x)^2}{6(a-x)(a+x)^2},\ \frac{30(a-x)}{6(a-x)(a+x)^2},$$
$$\frac{a+x}{6(a-x)(a+x)^2}.$$

[10.] The denominators of these fractions are,

$$2(x+1),\ x-1,\ 2(x+1)(x-1),\ 2,$$

the third of which contains all the others as factors; it is, therefore, the least common multiple of all; hence the numerator of the first fraction is to be multiplied by $x-1$, that of the second by $2(x+1)$, that of the third is to be left as it is, and that of the fourth is to be multiplied by $x^2-1$; and $2(x^2-1)$ will be the denominator common to all: so that the fractions are,

$$\frac{x-1}{2(x^2-1)},\ \frac{2(x+1)}{2(x^2-1)},\ \frac{1}{2(x^2-1)},\ \frac{x^2-1}{2(x^2-1)}.$$

[11.] Multiplying each numerator by the denominator of the other fraction, and putting the product of the denominators, namely $\{a-(b-c)\}\{a+(b-c)\}$, underneath, the changed fractions become

$$\frac{(a+b-c)^2}{a^2-(b-c)^2},\ \frac{(a-b+c)^2}{a^2-(b-c)^2}.$$

[12.] The denominators are $4(x^2-1)$, $8(x^2-2)$, $3(x^2-1)$, $6(x^2+2)(x^2-2)$: the least common multiple of these is

$8 \times 3(x^2-1)(x^2+2)(x^2-2)$ that is $24(x^2-1)(x^4-4)$; therefore the fractions in a common denominator are,

$$\frac{6(x^4-4)}{24(x^2-1)(x^4-4)},\quad \frac{3(x^2-1)(x^2+2)}{24(x^2-1)(x^4-4)},\quad \frac{8(x^4-4)}{24(x^2-1)(x^4-4)},$$

$$\frac{4(x^2-1)}{24(x^2-1)(x^4-4)}.$$

---

## ADDITION AND SUBTRACTION OF FRACTIONS.

### Page 87.

[1.] $\frac{3x}{2}+\frac{7x-4}{3}=\frac{9x+14x-8}{6}=\frac{23x-8}{6}$.

[2.] $3x+\frac{2x-5}{7}-\frac{1}{14}=\frac{42x+4x-10-1}{14}=\frac{46x-11}{14}$.

[3.] $\frac{x-y}{x+y}+\frac{x^2+y^2}{x^2-y^2}=\frac{(x-y)^2+x^2+y^2}{x^2-y^2}=\frac{2(x^2-xy+y^2)}{x^2-y^2}$.

[4.] $\frac{x-1}{2x}+\frac{2x+1}{3}-\frac{5}{6}=\frac{3x-3+4x^2+2x-5x}{6x}=\frac{4x^2-3}{6x}$

[5.] $\frac{2x-3}{4}-\frac{3x+4}{7}+\frac{5-2x}{28}=\frac{14x-21-12x-16+5-2x}{28}$

$$=-\frac{32}{28}=-\frac{8}{7}.$$

[6.] $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1-x}{4(1-x^2)}+\frac{1+x}{4(1-x^2)}+$
$\frac{1}{2(1+x^2)}=\frac{1}{2(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1+x^2+1-x^2}{2(1-x^4)}=\frac{1}{1-x^4}$.

[7.] $\frac{x^2-1}{8x+4}-x+\frac{1}{4}=\frac{x^2-1-8x^2-4x+2x+1}{8x+4}=-\frac{7x^2+2x}{8x+4}$.

[8.] $\frac{x}{2}-\frac{2x}{x+2}+\frac{x^3-2x^2}{2(4-x^2)}+1=\frac{x+2}{2}-\frac{2x}{x+2}-\frac{(x-2)x^2}{2(x^2-4)}=$

$$\frac{(x+2)^2-4x}{2(x+2)}-\frac{x^2}{2(x+2)}=\frac{4}{2(x+2)}=\frac{2}{x+2}.$$

[9.] $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}=\frac{2(x+1)-x+1}{2(x^2-1)}-$

$$\frac{x+3}{2(x^2+1)}=\frac{x+3}{2(x^2-1)}-\frac{x+3}{2(x^2+1)}=\frac{(x+3)(x^2+1)-(x+3)(x^2-1)}{2(x^4-1)}$$

$$=\frac{2(x-3)}{2(x^4-1)}=\frac{x-3}{x^4-1}.$$

[10.] $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{x(x+3)+x(x-3)}{x^2-9}-2$

$$=\frac{2x^2-2x^2+18}{x^2-9}=\frac{18}{x^2-9}.$$

[11.] $\frac{x^3}{x-1}-\frac{x^2}{x+1}-\frac{1}{x+1}+\frac{1}{x+1}=\frac{x^3(x+1)-x^2(x-1)}{x^2-1}=$

$$\frac{x^2(x^2+1)}{x^2-1}.$$

[12.] $\frac{x^3{}_n}{x^n-1}-\frac{x^{3n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}=\frac{x^{3n}-1}{x^n-1}-\frac{x^{2n}-1}{x^n+1}$,

and by actually dividing, this becomes,

$$x^{2n}+x^n+1-x^n+1=x^{2n}+2.$$

[13.] $\frac{1}{4a^2(x-a)}+\frac{1}{4a^2(x+a)}-\frac{x}{2a^2(x^2+a^2)}$

$$=\frac{x+a+x-a}{4a^2(x^2-a^2)}-\frac{x}{2a^2(x^2+a^2)}$$

$$=\frac{x}{2a^2(x^2-a^2)}-\frac{x}{2a^2(x^2+a^2)}=\frac{2a^2x}{2a^2(x^4-a^4)}=\frac{x}{x^4-a^4}.$$

[14.] $$\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}=\frac{a-x+a+x}{4a^3(a^2-x^2)}+$$
$$\frac{1}{2a^2(a^2+x^2)}=\frac{1}{2a^2(a^2-x^2)}+\frac{1}{2a^2(a^2+x^2)}=\frac{1}{a^4-x^4}.$$

[15.] $$\frac{x+y}{y}-\frac{2x}{x+y}+\frac{x^2(x-y)}{y(y^2-x^2)}=\frac{(x+y)^2-2xy}{y(x+y)}-\frac{x^2}{y(x+y)}$$
$$=\frac{y^2}{y(x+y)}=\frac{y}{x+y}.$$

[16.] $$\frac{1}{2(x+1)}+\frac{x+3}{2(x^2+1)}+\frac{x+3}{x^4-1}=\frac{x-1}{2(x^2-1)}+\frac{x+3}{2(x^2+1)}+$$
$$\frac{x+3}{x^4-1}=\frac{(x-1)(x^2+1)+(x+3)(x^2-1)}{2(x^4-1)}+\frac{x+3}{x^4-1}=$$
$$\frac{x^3+x^2-2+x+3}{x^4-1}=\frac{x^3+x^2+x+1}{x^4-1};$$
and by actually dividing numerator and denominator by the numerator, this becomes $\frac{1}{x-1}$.

[17.] Introducing the factors into the denominators of the second and third fractions, so as to make the denominators like that of the first, we have,
$$\frac{2x^2-9(x-1)(x-2)+8(x-1)(x-3)}{2(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}.$$
In this example the application of theorem 3, at page 44 of the Algebra, will save a good deal of work: thus, by aid of that theorem, we find at once $(x-1)(x-2)=x^2-3x+2$; and $(x-1)(x-3)=x^2-4x+3$: hence the numerator is $x^2-5x+6$; and we see, by the same theorem, that $(x-2)(x-3)$, in the denominator, is the same thing: consequently the fraction is reduced simply to $\frac{1}{2(x-1)}$.

[18.]

$$x-\frac{2x^2-\frac{1}{2}}{2x+1}-\frac{1}{2}=\frac{2x^2+x-2x^2+\frac{1}{2}}{2x+1}-\frac{1}{2}=\frac{x+\frac{1}{2}}{2x+1}-\frac{1}{2}=0.$$

[19.] $\frac{1}{8(x-1)}-\frac{1}{4(x-3)}+\frac{1}{8(x-5)}=$

$$\frac{(x-3)(x-5)-2(x-1)(x-5)+(x-1)(x-3)}{8(x-1)(x-3)(x-5)}=$$

$$\frac{x^2-8x+15-2(x^2-6x+5)+x^2-4x+3}{8(x-1)(x-3)(x-5)}=\frac{1}{(x-1)(x-3)(x-5)}.$$

[20.] $\frac{a}{a+b}+\frac{b}{a-b}=\frac{a^2+b^2}{a^2-b^2}$, and $\frac{a}{a-b}-\frac{b}{a+b}=\frac{a^2+b^2}{a^2-b^2}$; hence the expressions are equal.

[21.] $\frac{a}{a+c}-\frac{b}{b+c}=\frac{ab+ac-ab-bc}{(a+c)(b+c)}=\frac{ac-bc}{(a+c)(b+c)}$,

and $\frac{c}{b+c}-\frac{c}{a+c}=\frac{ac+c^2-bc-c^2}{(a+c)(b+c)}=\frac{ac-bc}{(a+c)(b+c)}$.

[22.] $\frac{b}{d}+\frac{ad-bc}{d(c-dx)}=\frac{bd(c-dx)+d(ad-bc)}{d^2(c-dx)}=\frac{a-bx}{c-dx}$,

and $\frac{a}{c}+\frac{(ad-bc)x}{c(c-dx)}=\frac{ac(c-dx)+(ad-bc)cx}{c^2(c-dx)}=\frac{a-bx}{c-dx}$.

## MULTIPLICATION AND DIVISION OF FRACTIONS.

### Pages 90, 91.

[1.] $\frac{a+x}{3}\times\frac{a-x}{2y}=\frac{a^2-x^2}{6y}$.

[2.]

$$\frac{(a+x)(a-x)}{b^2-y^2}\times\frac{b^2+y^2}{a+x}=\frac{(a-x)(b^2+y^2)}{b^2-y^2}=\frac{ab^2+ay^2-b^2x^2-xy^2}{b^2-y^2}.$$

[3.] $\frac{a^2+x^2}{ax}\times\frac{a^2-y^2}{ay}=\frac{a^4-a^2y^2+a^2x^2-x^2y^2}{a^2xy}$.

[4.]

$$\frac{a+x}{a}\cdot\frac{a-x}{a}\cdot\frac{-1}{a^2-x^2}=-\frac{1}{a^2}.$$

[5.]

$$\frac{(a+x)^2}{(a-y)^3}\cdot(a+x)^2(a-y)^3=(a+x)^4.$$

[6.] $\frac{x^4-y^4}{(x-y)^2}\cdot\frac{x-y}{x(x+y)}=\frac{(x^2+y^2)(x^2-y^2)}{x-y}\cdot\frac{1}{x(x+y)}=\frac{x^2+y^2}{x}.$

[7.] $\frac{a^2-x^2}{a}\cdot\frac{a^2+x^2}{ax}=\frac{a^4-x^4}{a^2x}.$ [8.] $\frac{a^4-x^4}{a^2x}\cdot\frac{ax}{a^2+x^2}=\frac{a^2-x^2}{a}.$

[9.] $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}=\left(\frac{x}{y}-\frac{y}{x}\right)^2$, and this, divided by $\frac{x}{y}-\frac{y}{x}$, gives $\frac{x}{y}-\frac{y}{x}$ for quotient.

[10.]

$$\frac{a+b}{1}\cdot\frac{x-y}{1}\cdot\frac{a^2}{x-y}=a^2(a+b).$$

[11.]

$$\frac{a^2}{a-x}\cdot\frac{a^2}{a+x}\cdot\frac{a^2-x^2}{a^4}=1.$$

[12.]

$$\frac{x^4-y^4}{(x-y)^2}\cdot\frac{x-y}{x(x+y)}=\frac{x^4-y^4}{x(x^2-y^2)}=\frac{x^2+y^2}{x}.$$

[13.]

$$\frac{(a^2+1)b}{1+ab}\cdot\frac{1+ab}{a^2+1}=b.$$

[14.] $\frac{(a^2-x^2)^2}{ax(a^2+x^2)}\cdot\frac{a^2+x^2}{a^2-x^2}=\frac{a^2-x^2}{ax}.$

[15.]

$$\begin{array}{l}
x+\tfrac{1}{2}y-2\\
\tfrac{1}{4}x+3y\\
\hline
\tfrac{1}{4}x^2+\tfrac{1}{8}xy-\tfrac{1}{2}x\\
\qquad 3xy+\tfrac{3}{2}y^2-6y\\
\hline
\underline{\tfrac{1}{4}x^2+\tfrac{25}{8}xy+\tfrac{3}{2}y^2-\tfrac{1}{2}x-6y}=\tfrac{1}{8}(2x^2+25xy+12y^2-4x-48y);
\end{array}$$

$$\text{or, }\begin{array}{l}
\tfrac{1}{2}(2x+y-4)\\
\tfrac{1}{4}(x+12y)\\
\hline
2x^2+xy-4x\\
\qquad 24xy+12y^2-48y\\
\hline
\tfrac{1}{8}(2x^2+25xy+12y^2-4x-48y)\\
\hline
\end{array}$$

[16.] $$\begin{array}{l} \tfrac{1}{2}x+\tfrac{1}{2})x^3+\tfrac{3}{4}x^2+\tfrac{3}{4}x+1\,(2x^2-\tfrac{1}{2}x+2=2(x^2-\tfrac{1}{2}x+1) \\ \qquad\quad \underline{x^3+\ x^2} \\ \qquad\qquad -\tfrac{1}{4}x^2+\tfrac{3}{4}x \\ \qquad\qquad \underline{-\tfrac{1}{4}x^2-\tfrac{1}{4}x\qquad} \\ \qquad\qquad\qquad\qquad\ x+1 \\ \qquad\qquad\qquad\qquad\ \underline{x+1} \end{array}$$

$$\begin{array}{l} \text{or, } x+1)4x^3+3x^2+3x+4(4x^2-x+4 \\ \qquad\qquad \underline{4x^3+4x^2\quad} \\ \qquad\qquad\quad -\ \ x^2+3x \\ \qquad\qquad\quad \underline{-\ \ x^2-\ \ x\quad} \\ \qquad\qquad\qquad\qquad\ 4x+4 \\ \qquad\qquad\qquad\qquad\ \underline{4x+4} \end{array}$$

As in the second mode of proceeding, we have multiplied the dividend by 4, and the divisor only by 2, the quotient must be double of the true quotient: hence, dividing by 2, we get $2x^2-\frac{1}{2}x+2$, as in the other method.

[17.] $$\frac{x^4+1+2x^3-2x}{x^2}\cdot\frac{x}{x^2-1+x}=\frac{x^4+2x^3-2x+1}{x(x^2+x-1)}.$$

[18.] $$\begin{array}{ll} & x^2+2x-1-\frac{2}{x}+\frac{1}{x^2}\ [x+1-\frac{1}{x} \\ & \underline{x^2\qquad\qquad} \\ 2x+1] & 2x-1 \\ & \underline{2x+1\qquad\qquad\qquad} \\ 2x+2-\frac{1}{x}] & \quad -2-\frac{2}{x}+\frac{1}{x^2} \\ & \quad \underline{-2-\frac{2}{x}+\frac{1}{x^2}}. \end{array}$$

[19.] $$\frac{a^2}{b^2}-2+\frac{b^2}{a^2}=\left(\frac{a}{b}-\frac{b}{a}\right)^2$$

therefore, the square root is, $\frac{a}{b}-\frac{b}{a}$.

[20.]
$$\begin{array}{r|l} & x^2+\frac{2ax}{3}-bx+\frac{a^2}{9}-\frac{ab}{3}+\frac{b^2}{4}\ \left[x+\frac{a}{3}-\frac{b}{2}\right. \\ & x^2 \\ \hline 2x+\frac{a}{3} & \frac{2ax}{3}-bx \\ & \frac{2ax}{3}+\frac{a^2}{9} \\ \hline 2x+\frac{2a}{3}-\frac{b}{2} & -bx-\frac{ab}{3}+\frac{b^2}{4} \\ & -bx-\frac{ab}{3}+\frac{b^2}{4} \\ \hline \end{array}$$

In this operation the fractions may be avoided by multiplying the proposed expression by 36; the square root will then be 6 times what it ought to be; so that, dividing it by 6, we shall get the same result as that here found.

[21.]
$$\begin{array}{r|l} x+a) & x \quad \left(1-\frac{a}{x}+\frac{a^2}{x^2}-\frac{a^3}{x^3}+\frac{a^4}{x^4}-\frac{a^5}{x^4(x+a)}\right. \\ & x+a \\ \hline & -a \\ & -a-\frac{a^2}{x} \\ \hline & \frac{a^2}{x} \\ & \frac{a^2}{x}+\frac{a^3}{x^2} \\ \hline & -\frac{a^3}{x^2} \\ & -\frac{a^3}{x^2}-\frac{a^4}{x^3} \\ \hline & \frac{a^4}{x^3} \\ & \frac{a^4}{x^3}+\frac{a^5}{x^4} \\ \hline & -\frac{a^5}{x^4} \end{array}$$

[22.]

$$b+x)\,a+x\,\left(\frac{a}{b}-\frac{(a-b)x}{b^2}+\frac{(a-b)x^2}{b^3}-\frac{(a-b)x^3}{b^4}+\frac{(a-b)x^4}{b^4(b+x)}\right.$$

$$\underline{a+\frac{ax}{b}}$$

$$-\frac{(a-b)x}{b}$$

$$\underline{-\frac{(a-b)x}{b}-\frac{(a-b)x^2}{b^2}}$$

$$\frac{(a-b)x^2}{b^2}$$

$$\underline{\frac{(a-b)x^2}{b^2}+\frac{(a-b)x^3}{b^3}}$$

$$-\frac{(a-b)x^3}{b^3}$$

$$\underline{-\frac{(a-b)x^3}{b^3}-\frac{(a-b)x^4}{b^4}}$$

$$\frac{(a-b)x^4}{b^4}.$$

[23.]

$$a^2+2ax+x^2)\,a^2\,\left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{(5a+4x)x^4}{a^3(a+x)^2}\right.$$

$$\underline{a^2+2ax+x^2}$$

$$-2ax-x^2$$

$$\underline{-2ax-4x^2-\frac{2x^3}{a}}$$

$$3x^2+\frac{2x^3}{a}$$

$$\underline{3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}}$$

$$-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$

$$\underline{-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}}$$

$$\frac{5x^4}{a^2}+\frac{4x^5}{a^3}.$$

[24.]
$$\begin{array}{rrrrl}
1-\frac{x}{2}+ & \frac{2x^2}{3}- & \frac{3x^3}{4} & +, & \&c. \\
1-\frac{x}{2}+ & \frac{2x^2}{3}- & \frac{3x^3}{4} & +, & \&c. \\
\hline
1-\frac{x}{2}+ & \frac{2x^2}{3}- & \frac{3x^3}{4} & +, & \&c. \\
-\frac{x}{2}+ & \frac{x^2}{4}- & \frac{x^3}{3} & +, & \&c. \\
 & \frac{2x^2}{3}- & \frac{x^3}{3} & +, & \&c. \\
 & - & \frac{3x^3}{4} & +, & \&c. \\
\hline
1-x+ & \frac{19}{12}x^2- & \frac{13}{6}x^3 & +, & \&c. \\
\hline
\end{array}$$

This 24th example might have been easily worked by the method of detached coefficients, explained at page 15 of this Key.

## GREATEST COMMON MEASURE, AND LEAST COMMON MULTIPLE.—PAGE 97.

[1.]
$$\begin{array}{l}
x^2+2x+1]\,x^3+2x^2+2x+1\,[x \\
\qquad\qquad\quad \underline{x^3+2x^2+\quad x} \\
\qquad\qquad\qquad\qquad\quad x+1]\,x^2+2x+1\,[x+1\,; \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^2+2x+1}
\end{array}$$

therefore the G. C. M. is $x+1$.

The two expressions are the same as these, viz.

$$(x^3+1)+2x(x+1) \text{ and}$$
$$(x+1)^2$$

in which $x+1$ is seen to be the greatest factor common to both.

[2.] Multiplying the first expression by 7 to prevent the entrance of fractions, the work is as follows:

$$\begin{array}{l}
7x^2-12x+5]\,14x^3+\ 7x^2-56x+35\,[2x \\
\qquad\qquad\quad \underline{14x^3-24x^2+10x} \\
\qquad\qquad\qquad\quad 31x^2-66x+35]\,7\times31x^2-372x+155\,[7 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{7\times31x^2-462x+245} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 90x-\ 90
\end{array}$$

$$\begin{array}{rrl}
\text{or, G. C. M.} & x-1\,] & 31x^2-66x+35\ [31x+35 \\
 & & \underline{31x^2-31x\phantom{+35}} \\
 & & \phantom{31x^2-}35x-35 \\
 & & \phantom{31x^2-}\underline{35x-35}
\end{array}$$

By the exercise of a little ingenuity, the proposed expressions might have been put in the following forms, viz.

$$\begin{gathered} 2x(x^2-1)+(x^2-1)-6(x-1), \\ 7(x^2-1)-12(x-1) \end{gathered}$$

where it is obvious that $x-1$ is the greatest factor common to both.

$$\begin{array}{rll}
[3.]\ 3x^2-2x-1\,] & 3\times4x^3-6x^2-9x+3\ [4x & \\
 & \underline{3\times4x^3-8x^2-4x\phantom{+3}} & \\
 & \phantom{3\times4x^3-}2x^2-5x+3\,] & 2\times3x^2-4x-2\ [3 \\
 & & \underline{2\times3x^2-15x+9} \\
 & & \phantom{2\times3x^2-}11x-11
\end{array}$$

$$\begin{array}{rrl}
\text{or, G. M. C.} & x-1\,] & 2x^2-5x+3\ [2x-3 \\
 & & \underline{2x^2-2x\phantom{+3}} \\
 & & \phantom{2x^2}-3x+3 \\
 & & \phantom{2x^2}\underline{-3x+3}
\end{array}$$

The expressions may be decomposed as follows, viz.

$$\begin{gathered} 4(x^3-1)-2(x^2-1)-3(x-1), \\ 3(x^2-1)-2(x-1) \end{gathered}$$

where the greatest factor common to both is evidently $x-1$.

[4.]

$$\begin{array}{rll}
12x^2-15x+3\,] & \phantom{\text{or, }}6x^3-6x^2+2x-2 & \\
\text{or, }4x^2-5x+1\,] & \text{or, }12x^3-12x^2+4x-4\ [3x & \\
 & \phantom{\text{or, }}\underline{12x^3-15x^2+3x\phantom{-4}} & \\
 & \phantom{\text{or, }12x^3-}3x^2+x-4\,] & 12x^2-15x+3\ [4 \\
 & & \underline{12x^2+4x-16} \\
 & & \phantom{12x^2}-19x+19
\end{array}$$

$$\begin{array}{rrl}
\text{or, G. C. M.} & x-1\,] & 3x^2+x-4\ [3x+4 \\
 & & \underline{3x^2-3x\phantom{-4}} \\
 & & \phantom{3x^2+}4x-4 \\
 & & \phantom{3x^2+}\underline{4x-4}
\end{array}$$

These expressions are easily decomposed: they are,

$$6x^2(x-1)+2(x-1)$$
$$\text{and } 12x(x-1)-3(x-);$$

where $x-1$ is at once seen to be, the greatest factor common to both.

(5.) Dividing by $3x$, the second expression gives $4x-16=(x^2+4)(x^2-4)$, and the first may be put in the form, $2x^2(x^2-4)-3(x^2-4)$, consequently the greatest common factor is $x^2-4$. From the work of the foregoing examples, it is evident that it is always desirable, unless the proposed expressions are very favourable for the operation, to dispense with the process by the rule, and to endeavour to accomplish the object of that process by simpler means. Instances are given above where the expressions may be so transformed as to exhibit the factor common to both with but little trouble: and the example next following will suggest to the learner another expedient that may often be applied with advantage.

[6.] The first expression here is, $(x^4-4x^3+4x^2)+(4x^2-16x+16)$; that is, $(x^2-2x)^2+(2x-4)^2=x^2(x-2)^2+4(x-2)^2=(x^2+4)(x-2)^2$. Consequently, if the two expressions have a factor in common, the second must be divisible either by $x^2+4$, or by $x-2$; probably even by $(x-2)^2$. Now, without putting this to the test, as respects the factor $x^2+4$, a glance at the second expression shows that no factor combined with this can produce that expression: for, to bring $-12x$ into the product of both, the second factor must obviously contain the term $-3x$; so that the term in the product involving $x^3$ would be $-3x^3$; whereas it ought to be $-6x^3$. The factor $x^2+4$ cannot, therefore, be common to both expressions: but the factor $x-2$ *is* common to both; inasmuch as the second becomes 0, for $x=2$. (See p. 55, Algebra.) It is worth while, therefore, to divide this second expression by $(x-2)^2$, or $x^2-4x+4$, and this we shall do by the method of detached coefficients, so often used in the former part of this Key: should there be a remainder, we shall then conclude that $x-2$ is the G. C. M.

$$\begin{array}{r} 1-4+4)\,1-6+13-12+4\,(1-2+1 \\ \underline{1-4+\phantom{1}4\phantom{-12+4}} \\ \phantom{1}-2+9-12\phantom{+4} \\ \underline{-2+8-\phantom{1}8\phantom{+4}} \\ 1-4+4 \\ \underline{1-4+4} \end{array}$$

As there is no remainder, we at once infer that $(x-2)^2$ is the G. C. M.

## LEAST COMMON MULTIPLE.

### PAGE 97.

[1.] THE proposed expressions are, $x^3+1$ and $(x+1)^2$. We know that the former is divisible by $x+1$ (Algebra, page 57); and as it is not divisible by this oftener than once, we conclude that $x+1$ is the G. C. M. of the two expressions, and that the product of both divided by $x+1$ is $(x^3+1)(x+1)=x^4+x^3+x+1$, the L. C. M.

[2.] Each of these expressions is divisible by $x-y$, but by no other factor; and the product of the two divided by $x-y$ is $(x^3-y^3)(x+y)=x^4+x^3y-xy^3-y^4$, the L. C. M.

[3.] These expressions may evidently be written in the following forms, namely:

$$2x(x^2-1)+(x^2-1),$$
$$2x(x^2-1)-(x^2-1) \quad ☞ \; [=(2x-1)(x^2-1)];$$

so that $x^2-1$ is their G. C. M.; consequently

$$(2x^3+x^2-2x-1)(2x-1)=4x^4-5x^2+1, \text{ the L. C. M.}$$

[4.] The G. C. M. of the first two of these is evidently $x-1$, and that of the last two $x-2$; so that the product of all is to be divided by $(x-1)(x-2)$; or rather, the factors $x-1$, $x-2$, are to be suppressed: we thus have $(x^2-1)(x^2-4)=x^4-5x^2+4$, the L. C. M.

In finding the product of these two factors, the learner will, of course, proceed upon the principle at page 44 of the Algebra.

[5.] These expressions are, $3x(x-a)$, $4x(x+a)$, $5(x^2-a^2)$, of which it is plain that the L. C. M. is, $3.4.5x(x^2-a^2)=60x(x^2-a^2)$.

## SIMPLE EQUATIONS:

### CLEARING FRACTIONS.—PAGE 106.

[1.] APPLYING the rule for reducing fractions to a common denominator, but omitting the denominator which equally divides all, we have,

$$\left.\begin{array}{rr} 2x\times7\times2\times5 = & 140x \\ -5x\times3\times2\times5 = & -150x \\ 1\times7\times3\times2\times5 = & 210 \\ 4\times3\times7\times2\times5 = & 840 \\ -3x\times3\times7\times5 = & -315x \\ x\times3\times7\times2 = & 42x \end{array}\right\}$$

Consequently, writing these results in order, the equation, cleared of fractions, is:

$$140x-150x+210$$
$$=840-315x+42x.$$

[2.] Multiplying each numerator by the product of the denominators of all the other fractions, we have,

$$15(x-1)+10(2x-5)=120x-6(x-2);$$
$$\text{or } 15x-15+20x-50=120x-6x+12.$$

[3] $d(ax-b)+2cd=c(bx-c)-3cd$; or $adx-bd+2cd=bcx-c^2-3cd.$

[4.] Here the denominators are removed by multiplying each side of the equation by 20, the least common multiple of the denominators: the result is,

$$4(2-3x)-2(x+3)=60+2x-1;$$
$$\text{or } 8-12x-2x-6=60+2x-1.$$

[5.] The least common multiple of the denominators is evidently $4bc^2$, therefore multiplying by this, the equation becomes $4c^2x-6cx+8bx=20bcx-3bc^2$.

[6.] The least common multiple of the denominators is $24x$, as is obvious; therefore, multiplying by this, the equation cleared of fractions is $192+56-48x=18-3+4x$.

[7.] Applying the rule for the common denominator, we have, $3(x+3)-14(x-1)+21=0$; or $3x+9-14x+14+21=0$..

[8.] Multiplying by 12, the least common multiple of 2, 3, 4, we have $4(a-\frac{1}{2})+6(x-\frac{1}{3})=27$; or $4x-2+6x-2=27$.

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## REMOVING RADICAL SIGNS.—PAGE 108.

[1.] Transposing, $4 = \sqrt{3 + 2x}$; squaring, $16 = 3 + 2x$.

[2.] Squaring, $x + 2\sqrt{x} + 1 = 6 + x$;
transposing, $2\sqrt{x} = 5$; therefore, squaring, $4x = 25$.

[3.] Cubing, $x + 2 = 3x - 5$.

[4.] Squaring, $x - 3 = \sqrt{(x^2 - 3x + 4)}$;
and squaring again, $x^2 - 6x + 9 = x^2 - 3x + 4$.

[5.] Transposing, $\sqrt[n]{(x + a)} = \sqrt[2n]{(x^2 + bx + c)}$;
raising each side to the power $2n$,
$(x + a)^2 = x^2 + bx + c$; or $x^2 + 2ax + a^2 = x^2 + bx + c$.
It is plain that the $n$th power of the first member is $x + a$, and as the $2n$th power is the $n$th power *squared*, the result must be as here stated.

[6.] Transposing, $\sqrt{(x + a)} = \sqrt{(x - \sqrt{b^2 + x^2})}$;
squaring, $x + a = x - \sqrt{b^2 + x^2}$;
transposing, $\sqrt{b^2 + x^2} = -a$
squaring, $b^2 + x^2 = a^2$.

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## SOLUTION OF SIMPLE EQUATIONS: FIRST SET.

### PAGE 114.

[1.] Transposing, $8x - 5x = 12 + 3$; that is, $3x = 15$;
$\therefore$ dividing each side by 3, we have, $x = 5$.

[2.] Removing the vinculum, $3x - 3 + 2 = 5x - 7$;
transposing, $-3 + 2 + 7 = 5x - 3x$, $\therefore 6 = 2x$, $\therefore x = 3$.

[3.] Removing vincula, $2x + 6 - 6 = 6 - 12x + 92$;
transposing, $2x + 12x = 6 + 92$; that is, $14x = 98$, $\therefore x = 7$.

[4.] Removing vincula, $14x - 6x + 10 = 3x + 30$;
transposing, $14x - 6x - 3x = 30 - 10$;
that is, $5x = 20$, $\therefore x = 4$.

[5.] Removing vincula, $6x + 8 - 6x + 3 + 1 = 3x - 15$; transposing, $8 + 3 + 1 + 15 = 3x$, that is, $27 = 3x$, $\therefore x = 9$.

[6.] $20x - 12 - 18x + 3 = 2 - 14x + 6x$;
transposing, $20x - 18x + 14x - 6x = 2 + 12 - 3$,
that is, by collecting, $10x = 11$, $\therefore x = \frac{11}{10} = 1\frac{1}{10}$.

[7.] $6 - 4x + 15x - 30 = 18x - 3 + 8$;
transposing, $-4x + 15x - 18x = -3 + 8 - 6 + 30$,
that is, $-7x = 29$, $\therefore x = -\frac{29}{7} = -4\frac{1}{7}$.

[8.] By theorem 3, page 44, Algebra, $x^2 + x - 2 = x^2 + 2x - 15$. expunging $x^2$, and transposing, $15 - 2 = 2x - x$, $\therefore x = 13$.

[9.] By actually multiplying,

$$12x^2 - 13x + 3 = 12x^2 + 16x - 35;$$

expunging $12x^2$, and transposing, $35 + 3 = 16x + 13x$;
that is, $38 = 29x$, $\therefore x = \frac{38}{29} = 1\frac{9}{29}$.

[10.] Transposing, $(a - c)x = d - b$, $\therefore x = \dfrac{d - b}{a - c}$.

[11.] By theorem 3, page 44, $x^2 + 2x - 15 = x^2 + 4x + 4$,
expunging $x^2$, and transposing, $2x - 4x = 4 + 15$;
that is, $-2x = 19$, $\therefore x = -\frac{19}{2} = -9\frac{1}{2}$.

[12.] Transposing, $(a-b)x = a^2 - b^2$, $\therefore x = \dfrac{a^2 - b^2}{a - b} = a + b$.

[13.] $3a - 3x + 2 = 10x - 5a$,
transposing, $3a + 5a + 2 = 10x + 3x$;
that is, $8a + 2 = 13x$, $\therefore x = \frac{1}{13}(8a + 2) = \frac{2}{13}(4a + 1)$.

[14.] $ax - ab^2 = cx + cb$, $\therefore$
transposing, $(a - c)x = (ab + c)b$, $\therefore x = \dfrac{(ab + c)b}{a - c}$.

[15.] By theorem 3, page 44, $x^2 + (a+b)x + ab - ab - a = x^2$;
that is, $(a+b)x = a$, $\therefore x = \dfrac{a}{a + b}$.

[16] By theorem 3, page 44,

$$x^2 + 34x + 168 - x^2 - 42x - 152 = 0;$$

transposing, $168 - 152 = 42x - 34x$;
that is, $16 = 8x$, $\therefore x = 2$.

## SECOND SET.—PAGES 114, 115.

[17.] Multiplying by 4, $4x-6x=x-12$, $\therefore -2x=x-12$;
transposing, $12=3x$, $\therefore x=4$.

[18.] $9x-30x+8=24x-18$, $\therefore -21x+8=24x-18$;
transposing, $18+8=24x+21x$, $\therefore 26=45x$, $\therefore x=\frac{26}{45}$.

[19.] $2(2x+1)+40=5(3x-1)-30$
or $4x+2+40=15x-5-30$;
transposing, $2+40+5+30=15x-4x$;
that is, $77=11x$, $\therefore x=7$.

[20.] $6x-10x-110+15x=0$, $\therefore 11x=110$, $\therefore x=10$.

[21.] $52x-26-33x+55=143$, $\therefore 52x-33x=143+26-55$;
that is, $19x=114$, $\therefore x=6$.

[22.] Multiplying by 12, $6x+4x=84+3x$;
transposing, $6x+4x-3x=84$, $\therefore 7x=84$, $\therefore x=12$.

[23.] Multiplying by 6. $3x-9+4x+2+36=x-18x+18$;
transposing, $3x+4x-x+18x=18+9-2-36$;
that is, $24x=-11$, $\therefore x=-\frac{11}{24}$.

[24.] $15x-5=20x-8x+4$, $\therefore$
transposing, $15x-20x+8x=4+5$;
that is, $3x=9$ $\therefore x=3$.

[25.] Multiplying by 20,
$10x+30+20+4x-3x+1=0$;
transposing, $10x+4x-3x=-30-20-1$;
that is, $11x=51$, $\therefore x=-\frac{51}{11}=-4\frac{7}{11}$.

[26.] $63x+49-14x+2x-4=504$;
transposing, $63x-14x+2x=504-49+4$;
that is, $51x=459$, $\therefore x=9$.

[27.] $bcdx-acdx=abdx-abcx$,
$\therefore (abc+bcd-acd-abd)x=0$, $\therefore x=0$.

The learner will perceive that we are justified here in inferring that $x=0$, provided the co-efficient of $x$ is *anything*, for if two factors multiplied together produce 0, one of them at least must itself be 0.

[28.] $\frac{x^2-4}{x+2}-\frac{x^2-1}{x-1}=\frac{3x}{x-2}$ is obviously the same as $x-2-(x+1)=\frac{3x}{x-2}$; so that multiplying by $x-2$ to clear the fraction, we have $x^2-4x+4-x^2+x+2=3x$; that is, by transposing, $4+2=3x+4x-x$; or $6=6x$, $\therefore x=1$.

[29.] Multiplying by $\sqrt{x}$, $x-2=\frac{2x}{3}$, $\therefore 3x-6=2x$, $\therefore x=6$.

[30.] Multiplying by $a^2-x^2$, the L. C. M. of the denominators, we have $1-a^3+ax^2=ax(a+x)+a(a-x)$;
that is, $1-a^3+ax^2=a^2x+ax^2+a^2-ax$;
and transposing,

$$1-a^3-a^2=a^2x-ax=(a^2-a)x, \therefore x=\frac{1-a^3-a^2}{a^2-a}.$$

[31.] Multiplying by $5\times 3$, or 15, the L. C. M. of the denominators, we have $51-9x-20x-10=145-55x$;
transposing, $55x-9x-20x=145-51+10$;
that is, $26x=104, \therefore x=4$.

[32.] Multiplying by $4x(x+2)$, we have

$$8x+x^2(x+2)=(x^2+1)(x+2),$$

that is, by subtracting $x^2(x+2)$ from each side,

$$8x=x+2, \therefore 7x=2, \therefore x=\tfrac{2}{7}.$$

[33.] Multiplying by 60, the L. C. M. of the denominators, we have $30x+20x+15x+12x=10x+2220$;

$$\therefore 77x-10x=2220; \text{ that is, } 67x=2220,$$

$$\therefore x=\tfrac{2220}{67}=33\tfrac{9}{67}.$$

[34.] Multiplying by $11\times 8$,

$$88x-12x+\tfrac{8}{5}(x-4)-8-\tfrac{33}{4}x+22=19\times 88;$$

that is, $76x+\frac{8}{5}(x-4)-\frac{33}{4}x=19\times 88-14=1658$.
Multiplying now by $4\times 5$, we have

$$1520x+32x-128-165x=33160;$$

or transposing and collecting,

$$1387x=33288, \therefore x=\tfrac{33288}{1387}=24.$$

[35.] Multiplying first by 4, the equation becomes

$$\frac{4x+3}{3}-\frac{43-4x}{3\frac{2}{3}}=1;\ \text{that is, } \frac{4x}{3}-\frac{129-12x}{11}=0,$$

the terms of the second fraction being multiplied by 3.

Clearing fractions, by the common rule,

$$44x-387+36x=0;$$

transposing, $80x=387$, $\therefore x=\frac{387}{80}=4\frac{67}{80}$.

[36.] By theorem 1, page 44, Algebra,

$$\frac{5x^2-9}{x\sqrt{5}+3}=\tfrac{1}{2}(x\sqrt{5}-3)=1,\ \therefore \text{the equation is,}$$

$$x\sqrt{5}-3-\tfrac{1}{2}(x\sqrt{5}-3)=1;\ \text{that is, } \tfrac{1}{2}(x\sqrt{5}-3)=1,$$

$$\therefore x\sqrt{5}-3=2,\ \therefore x\sqrt{5}=5,\ \therefore x=\frac{5}{\sqrt{5}}=\sqrt{5}.$$

[37.] Multiplying by 22, we have,

$$\tfrac{11}{2}\{3x-3-\tfrac{2}{3}x\}+2-\tfrac{4}{5}x=24+\tfrac{2}{5}x-\tfrac{4}{5};$$

that is, by transposing, &c. $\frac{11}{2}(\frac{7}{3}x-3)-\frac{6}{5}x=22-\frac{4}{5}$;
or, multiplying by 10, $55(\frac{7}{3}x-3)-12x=220-8=212$,
$\therefore$ multiplying by 3, $55(7x-9)-36x=636$;
that is, $885x-36x=636+495$,

$$\therefore 349x=1131,\ \therefore x=\tfrac{1131}{349}=3\tfrac{84}{349}.$$

[38.] Multiplying by 30, the L. C. M. of the denominators,

$$\begin{aligned}12x^2-6x-18-3&=2(2x-1)(3x+2)\\&=12x^2+2x-4;\end{aligned}$$

that is, $-6x-21=2x-4$, $\therefore -17=8x$, $\therefore x=-\frac{17}{8}=-2\frac{1}{8}$.

[39.] This equation is the same as

$$x^2+(a+b)x+ab+\frac{a}{x}+\frac{x}{a}=x^2+\frac{x}{a}+\frac{a^2+b^2}{b}+ab+\frac{a}{x}:$$

that is, by cancelling equal terms on both sides,

$$(a+b)x=\frac{a^2+b^2}{b},\quad \therefore x=\frac{a^2+b^2}{(a+b)b}.$$

## THIRD SET.—PAGES 115, 116.

[40.] Transposing, $\sqrt{(2x+x^2)} = 1 - x$;
squaring, $2x + x^2 = 1 - 2x + x^2$; $\therefore 4x = 1$, $\therefore x = \frac{1}{4}$.

[41.] Transposing, $\sqrt{(3+4x)} = 2x + 1$;
$\therefore 3 + 4x = 4x^2 + 4x + 1$; $\therefore 4x^2 = 2$; $\therefore x^2 = \frac{1}{2}$; $\therefore x = \sqrt{\frac{1}{2}}$.

[42.] $\sqrt{(a+x)} = b - a$; $\therefore a + x = (b-a)^2$; $\therefore x = (b-a)^2 - a$.

[43.] By squaring, $x + 7 = x + 2\sqrt{x} + 1$;
transposing, $6 = 2\sqrt{x}$; $\therefore \sqrt{x} = 3$; $\therefore x = 9$.

[44.] Transposing, $\sqrt{(4x+9)} = 2\sqrt{x} + 1$;
squaring, $4x + 9 = 4x + 4\sqrt{x} + 1$;
$\therefore 8 = 4\sqrt{x}$, $\therefore \sqrt{x} = 2$, $\therefore x = 4$.

[45.] Transposing, $\sqrt{(x+8)} = \sqrt{(x-3)} + 1$;
squaring, $x + 8 = x - 3 + 2\sqrt{(x-3)} + 1$;
$\therefore$ transposing, $10 = 2\sqrt{(x-3)}$; $\therefore 5 = \sqrt{(x-3)}$;
squaring, $25 = x - 3$; $\therefore x = 28$.

[46.] Transposing, $\sqrt{(x-7)} = 13 - \sqrt{(x+6)}$;
squaring, $x - 7 = 169 - 26\sqrt{(x+6)} + x + 6$;
transposing, $26\sqrt{(x+6)} = 182$; $\therefore \sqrt{(x+6)} = 7$;
squaring, $x + 6 = 49$; $\therefore x = 43$.

[47.] Transposing, $\sqrt{(2ax + \frac{1}{4}x^2)} = 2a - \frac{1}{2}x$;
squaring, $2ax + \frac{1}{4}x^2 = 4a^2 - 2ax + \frac{1}{4}x^2$;
transposing, $4ax = 4a^2$; $\therefore x = a$.

[48.] Multiplying by $\sqrt{(x+1)}$, the equation becomes
$x + 1 + \sqrt{(x^2-1)} = 2$; $\therefore \sqrt{(x^2-1)} = 1 - x$;
squaring, $x^2 - 1 = (1-x)^2 = 1 - 2x + x^2$, $\therefore 2x = 2$, $\therefore x = 1$.

[49.] Squaring, $a + x - 2\sqrt{(ax+x^2)} + x = b$;
transposing, $2\sqrt{(ax+x^2)} = 2x + a - b$;
squaring, $4(ax+x^2) = 4x^2 + 4(a-b)x + (a-b)^2$;
transposing, $4bx = (a-b)^2$; $\therefore x = \dfrac{(a-b)^2}{4b}$.

[50.] Multiplying by $\sqrt{(a+x)}$, we have
$\sqrt{(ax+x^2)} + a + x = 2a$; $\therefore \sqrt{(ax+x^2)} = a - x$;
squaring, $ax + x^2 = a^2 - 2ax + x^2$; $\therefore 3ax = a^2$; $\therefore x = \dfrac{a}{3}$.

[51.] Squaring, $a+x+2\sqrt{(a^2-x^2)}+a-x=ax$;
that is, $2\sqrt{(a^2-x^2)}=ax-2a$;
squaring, $4(a^2-x^2)=a^2x^2-4a^2x+4a^2$;
$\therefore -4x^2=a^2x^2-4a^2x$; $\therefore -4x=a^2x-4a^2$;

$$\therefore 4a^2=(a^2+4)x;\ \therefore x=\frac{4a^2}{a^2+4}.$$

[52.] Squaring, $\dfrac{x+1}{x-1}+2+\dfrac{x-1}{x+1}=a^2$;

that is, $\dfrac{3x-1}{x-1}+\dfrac{x-1}{x+1}=a^2$;

clearing, $(x+1)(3x-1)+(x-1)^2=a^2(x^2-1)$;
that is, $4x^2=a^2x^2-a^2$; $\therefore a^2=(a^2-4)x^2$;

$$\therefore x^2=\frac{a^2}{a^2-4};\ \therefore x=\frac{a}{\sqrt{(a^2-4)}}.$$

[53.] Squaring, $\dfrac{1}{x}+1-2\left(\dfrac{1}{x^2}-1\right)^{\frac{1}{2}}+\dfrac{1}{x}-1=1$;

that is, $-2\left(\dfrac{1}{x^2}-1\right)^{\frac{1}{2}}=1+\dfrac{2}{x}$;

or, multiplying by $x$, $-2(1-x^2)^{\frac{1}{2}}=x+2$;
squaring, $4(1-x^2)=x^2+4x+4$;

transposing, $0=5x^2-4x$; $\therefore 5x-4=0$; $\therefore x=\dfrac{4}{5}$.

[54.] Squaring, $4+x\sqrt{(x^2-1)}=x^2-4x+4$;
$\therefore \sqrt{(x^2-1)}=x-4$; $\therefore x^2-1=x^2-8x+16$;
$\therefore 8x=17$; $\therefore x=2\frac{1}{8}$.

[55.] Raising each side to the power $2m$,
$(a+x)^2=x^2+5ax+b^2$;
that is, $a^2+2ax+x^2=x^2+5ax+b^2$;

$$\therefore a^2-b^2=3ax;\ \therefore x=\frac{a^2-b^2}{3a}.$$

[56.] Transposing and squaring,
$a^2-x\sqrt{(x^2+a^2)}=a^2-2ax+x^2$;
$\therefore -x\sqrt{(x^2+a^2)}=x^2-2ax$; $\therefore -\sqrt{(x^2+a^2)}=x-2a$;
squaring, $x^2+a^2=x^2-4ax+4a^2$;

$$\therefore 4ax=3a^2;\ \therefore x=\frac{3a}{4}.$$

[57.] Multiplying by 2, $2x\sqrt{(3+2x^2)} = 3+2x^2$;
dividing by $\sqrt{(3+2x^2)}$, $2x = \sqrt{(3+2x^2)}$;
squaring, $4x^2 = 3+2x^2$; $\therefore 2x^2 = 3$; $\therefore x = \sqrt{\frac{3}{2}}$.

[58.] Squaring, $ax - x^2 = b^2 - x^2$; $\therefore ax = b^2$; $\therefore x = \frac{b^2}{a}$.

[59.] Multiplying by $\sqrt{x}$, $x\sqrt{(6-x^2)} = 3+x^2$;
squaring, $x^2(6-x^2) = 9+6x^2+x^4$;
$\therefore -9 = 2x^4$; $\therefore x^4 = -4\frac{1}{2}$; $\therefore x = \sqrt[4]{-4\frac{1}{2}}$.

[60.] By the principle at p. 113, Algebra,

$$\sqrt{\frac{a+x}{a-a}} = \frac{\sqrt{b}+1}{\sqrt{b}-1};$$

squaring, $\frac{a+x}{a-x} = \frac{b+2\sqrt{b}+1}{b-2\sqrt{b}+1}$;

$\therefore (a+x)(b-2\sqrt{b}+1) = (a-x)(b+2\sqrt{b}+1)$;

transposing, $2(b+1)x = 4a\sqrt{b}$; $\therefore x = \frac{2a\sqrt{b}}{b+1}$.

[61.] Multiplying by 6, to clear fractions,
$4x\sqrt{(3+2x^2)} = 3+4x^2\sqrt{2}$;
squaring, $16x^2(3+2x^2) = 9+24x^2\sqrt{2}+32x^4$;
$\therefore 48x^2 = 9+24x^2\sqrt{2}$; $\therefore 4(12-6\sqrt{2})x^2 = 9$;

$\therefore x^2 = \frac{9}{4(12-6\sqrt{2})}$; $\therefore x = \frac{3}{2\sqrt{(12-6\sqrt{2})}}$.

[62.] Squaring,
$a^2-x^2+2x\sqrt{\{(a^2-x^2)(a^2-1)\}}+(a^2-1)x^2 = a^2(1-x^2)$;
transposing, $2x\sqrt{\{(a^2-x^2)(a^2-1)\}} = 2x^2-2a^2x^2$;
$\therefore \sqrt{\{(a^2-x^2)(a^2-1)\}} = (1-a^2)x$.

Or, dividing by $a^2-1$,

$$\sqrt{\frac{a^2-x^2}{a^2-1}} = -x;$$

squaring, $\frac{a^2-x^2}{a^2-1} = x^2$; $\therefore a^2-x^2 = (a^2-1)x^2$;
$\therefore a^2 = a^2x^2$; $\therefore x^2 = 1$; $\therefore x = \pm 1$. (See p. 65, Algebra.)

It is $x = -1$ that satisfies the original equation.

A good opportunity offers here to draw the attention of the learner to the fact, that when the proposed is an *irrational* equation, that is, an equation in which the unknown quantity is under the radical sign, the resulting values of that unknown quantity, as derivable from the *rational* equation deduced from the given one, will not *all of them* necessarily satisfy the original equation. It is not to be expected that such should be the case, because, in clearing the radicals, the rational equation obtained remains the same, whether those radicals have a plus sign or a minus sign; so that the results necessarily satisfy the proposed equation only, when the radicals in it are unrestricted as to sign. In the equation solved above, the radicals are assumed to be *plus:* after squaring, the radical in the second term, being united to the preceding term by +, is regarded as *positive,* although the equation $a^2-x^2=(a^2-1)x^2$ would equally be the result if the sign were *negative;* and it is *this* equation which is ultimately solved. It is in consequence of the restriction in the original equation, thus inoperative in the equation actually solved, that the number of solutions admissible into the former equation is often limited, though they all apply without limitation to the latter. The values $x=\pm 1$ *both* satisfy the rational equation solved; but only $x=-1$ satisfies the irrational equation at the commencement of the work.

[63.] Adding 1 to each side we have,

$$\frac{a^2+2ax+x^2}{a^2}=b^2;\ \text{that is, } \left(\frac{a+x}{a}\right)^2=b^2;$$

$$\therefore\ \frac{a+x}{a}=b;\ \therefore a+x=ab;\ \therefore x=a(b-1).$$

[64.] Multiplying by the denominator,

$$81\sqrt{3}=(4\sqrt{x}-\sqrt{3})^3;$$

taking the cube root of each side, $3\sqrt{3}=4\sqrt{x}-\sqrt{3};$

$$\therefore 4\sqrt{3}=4\sqrt{x};\ \therefore\ \sqrt{x}=\sqrt{3};\ \therefore x=3.$$

[65.] Adding 1 to each side, the equation becomes

$$\frac{1}{1-2x+x^2}=a+1;\ \text{that is, } \frac{1}{(1-x)^2}=a+1;$$

$$\therefore (1-x)^2=\frac{1}{a+1};\therefore 1-x=\frac{1}{\sqrt{(a+1)}};\therefore x=1-\frac{1}{\sqrt{(a+1)}}.$$

[66.] Squaring, $\frac{2ax-x^2}{(a-x)^2}=b^2; \therefore \frac{2ax-x^2}{(a-x)^2}+1=b^2+1;$

that is, $\frac{a^2}{(a-x)^2}=b^2+1$; or, $\frac{(a-x)^2}{a^2}=\frac{1}{b^2+1}$;

and taking the square root,

$$\frac{a-x}{a}=\sqrt{\frac{1}{b^2+1}}; \therefore x=a\left(1-\sqrt{\frac{1}{b^2+1}}\right).$$

[67.] This equation is, $x^2+\frac{1}{x^2}=x^2-2+\frac{1}{x^2}+\sqrt{x}$;

$$\therefore 2=\sqrt{x}; \therefore x=4.$$

[68.] Multiplying by $\sqrt{(a+x)}$ we have,

$$\sqrt{(ax+x^2)}+a+x=a^2+a;$$

$$\therefore \sqrt{(ax+x^2)}=a^2-x; \therefore ax+x^2=a^4-2a^2x+x^2;$$

$$\therefore (2a+1)ax=a^4; \therefore x=\frac{a^3}{2a+1}.$$

[69.] By the principle at page 113, Algebra, we get

$$\frac{a}{\sqrt{\left(\frac{2a}{x}-\frac{1}{x^2}\right)}}=\frac{1+b^2}{1-b^2};$$

squaring,

$$\frac{a^2}{\frac{2a}{x}-\frac{1}{x^2}}=\left(\frac{1+b^2}{1-b^2}\right)^2;$$

or, inverting the terms of each fraction,

$$\frac{\frac{2a}{x}-\frac{1}{x^2}}{a^2}=\left(\frac{1-b^2}{1+b^2}\right)^2;$$

subtract now each side from 1, and there results

$$\frac{a^2-\frac{2a}{x}+\frac{1}{x^2}}{a^2}=1-\left(\frac{1-b^2}{1+b^2}\right)^2=\left(\frac{2b}{1+b^2}\right)^2;$$

taking the square root of each side,

$$\frac{a-\frac{1}{x}}{a}=\frac{2b}{1+b^2}=\frac{ax-1}{ax};$$

$$\therefore 2abx=a(1+b^2)x-(1+b^2);$$

$$\therefore a\{2b-(1+b^2)\}x=-(1+b^2);\ \therefore x=\frac{1+b^2}{a(1-b)^2}.$$

[70.] This equation is, $1+\frac{\sqrt{(2ax-x^2)}}{a-x}=b;$

$$\therefore \frac{\sqrt{(2ax-x^2)}}{a-x}=b-1;$$

squaring, $\frac{2ax-x^2}{(a-x)^2}=(b-1)^2;$

adding 1 to each side, $\frac{a^2}{(a-x)^2}=(b-1)^2+1;$

and extracting the square root, we have

$$\frac{a}{a-x}=\sqrt{\{(b-1)^2+1\}};\ \therefore a=(a-x)\sqrt{\{(b-1)^2+1\}};$$

$$\therefore x=a\left(1-\sqrt{\frac{1}{(b-1)^2+1}}\right).$$

[71.] By the principle at page 113, Algebra,

$$\frac{\sqrt{(2x+x^2)}}{1+x}=\frac{3}{5};\ \therefore \frac{2x+x^2}{(1+x)^2}=\frac{9}{25};$$

subtracting each side from 1, we have

$$\frac{1}{(1+x)^2}=\frac{16}{25};\ \therefore \frac{1}{1+x}=\frac{4}{5};\ \therefore 5=4+4x;\ \therefore x=\frac{1}{4}.$$

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## Application of Simple Equations with one unknown Quantity to the Solution of Questions.—Page 124.

[1.] Let $x$ represent the number; then the double of it, diminished by 3, is expressed by $2x-3$; and the number itself increased by 8, by $x+8$; we therefore have the equation, $2x-3=x+8$; $\therefore$ transposing, $x=8+3=11$.

[2.] Let $x$ be the number; then by the question,

$$2x - \frac{x}{3} = 30, \text{ or clearing the fraction, } 6x - x = 90;$$

$$\therefore 5x = 90; \therefore x = 18.$$

We might have avoided the fraction thus:
let $3x$ be the number; then by the question,

$$6x - x = 30; \therefore 5x = 30; \therefore x = 6; \therefore 3x = 18.$$

[3.] Let $x$ be the number; then by the question,

$$\frac{x}{3} - \frac{x}{4} = 6, \text{ or clearing fractions, } 4x - 3x = 72;$$

$$\text{that is, } \quad x = 72.$$

Fractions might have been avoided thus:
let $12x$ be the number; then by the question,

$$4x - 3x = 6; \text{ that is, } x = 6; \therefore 12x = 72.$$

[4.] Suppose they meet in $x$ days; then by the question, one has travelled $16x$ miles and the other $18x$; and since these distances make up the whole 204 miles, we have the equation,

$$16x + 18x = 204; \text{ that is, } 34x = 204; \therefore x = \tfrac{204}{34} = 6,$$

the number of days.

[5.] Suppose the first has travelled $x$ miles; then the second must have travelled $153 - x$ miles; and as the *time* occupied by each is the same, there must be as many times 4 miles in $x$ miles as there are $3\frac{1}{2}$ miles in $153 - x$ miles: hence we have the equation,

$$\frac{x}{4} = \frac{153 - x}{3\frac{1}{2}}; \text{ or } \frac{x}{4} = \frac{306 - 2x}{7};$$

clearing fractions, $7x = 1224 - 8x$; $\therefore 15x = 1224$;

$$\therefore x = \tfrac{1224}{15} = \tfrac{408}{5} = 81\tfrac{3}{5}; \therefore 153 - x = 71\tfrac{2}{5};$$

consequently the first has travelled $81\frac{3}{5}$ miles, and the second $71\frac{2}{5}$ miles.

*Otherwise*, suppose they meet in $x$ hours, travelling at the rate of 4 miles and $3\frac{1}{2}$ miles an hour respectively, then the first will have travelled $4x$ miles, and the second $\frac{7x}{2}$ miles:

hence, by the question,

$$4x + \frac{7x}{2} = 153;\ \therefore 8x + 7x = 306, \text{ or } 15x = 306;$$

$\therefore x = \frac{306}{15} = \frac{102}{5} =$ the number of hours at 4 miles an hour: consequently the distance travelled by the first must be $\frac{102}{5} \times 4 = \frac{408}{5} = 81\frac{3}{5}$ miles, and $\therefore 153 - 81\frac{3}{5} = 71\frac{2}{5}$, the number of miles travelled by the second.

[6.] Let $x$ be the number; then by the question,
$\frac{3x}{4} + 2 = \frac{5x}{6} - 2$; or, multiplying by 12, $9x + 24 = 10x - 24$;
transposing, $48 = x$, the number required.

Fractions might have been avoided thus:
let $12x$ be the number; then by the question,

$$9x + 2 = 10x - 2;\ \therefore 4 = x;\ \therefore 12x = 48.$$

[7.] Suppose the third receives $x$ pounds,
then the second receives $x + 10$
and the first, $x + 18$
$\therefore \overline{3x + 28} = 110;$
transposing, $3x = 110 - 28 = 82$; $\therefore$ £$x$ = £$\frac{82}{3}$ = £27 6*s.* 8*d.*, the share of the third; $\therefore$ the share of the second is £37 6*s.* 8*d.*; and that of the first, £45 6*s.* 8*d.*

[8.] Suppose the third receives $x$ pounds; then
the second receives $2x$
and the first, $4x$
$\therefore \overline{7x} = 700;\ \therefore x = 100;$
hence the third receives £100, the second £200, and the first £400.

[9.] Suppose C receives $x$ pounds; then
B receives $3x$
and A receives $6x$
$\therefore 10x = 143;\ \therefore$ £$x$ = £$\frac{143}{10}$ = £14 6*s*;
hence C receives £14 6*s.* B, £42 18*s.* and A, £85 16*s.*

[10.] Suppose the third part is $x$ feet; then
the second part must be $x + 5$
and the first, $x + 2$
$\therefore \overline{3x + 7} = 37;\ \therefore 3x = 30;\ \therefore x = 10;$
hence the third part is 10 feet, the second 15 feet, and the first 12 feet.

[11.] Let the less number be $x$, then $x+6$ is the greater; and by the question, $\frac{x}{3}+\frac{x+6}{5}=\frac{x+6}{3}-\frac{x}{5}$;

multiplying by 15 to clear fractions,

$$5x+3x+18=5x+30-3x;$$

transposing, $6x=30-18=12$; $\therefore x=2$; $\therefore x+6=8$; hence the numbers are 2 and 8.

[12.] Suppose B can do it in $x$ days, then he can do $\frac{1}{x}$ of it in 1 day; also A can do $\frac{1}{9}$ of it in 1 day; and since A and B together can do $\frac{1}{6}$ of it in 1 day, it follows that $\frac{1}{6}-\frac{1}{9}$ is the part B alone can do in one day: but this part is also $\frac{1}{x}$; therefore

$$\frac{1}{6}-\frac{1}{9}=\frac{1}{x}, \text{ that is } \frac{1}{18}=\frac{1}{x}; \therefore x=18;$$

so that B alone can do it in 18 days.

[13.] Suppose they march $x$ hours, then the first detachment must have marched $x+2$ hours; the distance marched by the second detachment, at $3\frac{1}{2}$ or $\frac{7}{2}$ miles an hour, is therefore $\frac{7x}{2}$; and the distance marched by the first, at $2\frac{1}{2}$ or $\frac{5}{2}$ miles an hour, is $\frac{5(x+2)}{2}$: but these distances are the same; therefore

$$\frac{7x}{2}=\frac{5(x\times 2)}{2}; \therefore 7x=5x+10; \therefore 2x=10; \therefore x=5;$$

hence the second detachment marched 5 hours, the distance being $\frac{7+5}{2}=\frac{35}{2}=17\frac{1}{2}$ miles.

[14.] Suppose he had $x$ shillings at first, then after his first [illegible] he had $\frac{4x}{5}$, and winning 10*s.* he had $\frac{4x}{5}+10$; as he then lost $\frac{1}{3}$ of this, he had $\frac{2}{3}$ of it left, which, with the 3*s.* won, made his number of shillings $\frac{2}{3}\left(\frac{4x}{5}+10\right)+3$, which, by the question, amounted to 63 shillings;

$$\therefore \frac{2}{3}\left(\frac{4x}{5}+10\right)+3= 63; \quad \therefore \frac{2}{3}\left(\frac{4x}{5}+10\right)= 60;$$

$$\therefore 2\left(\frac{4x}{5}+10\right)=180, \quad \text{that is } \frac{8x}{5}+20=180;$$

$$\therefore \frac{8x}{5}=160; \therefore 8x=800; \therefore x=100;$$

hence he began with 100*s.* or £5.

*Otherwise,* suppose he had $5x$ shillings at first, then he lost $x$ shillings; so that, after winning 10*s.*, his number of shillings was $4x+10$: of this sum he had afterwards but $\frac{2}{3}$ds and 3*s.*, which by the question was 63*s.*;

$$\therefore \tfrac{2}{3}(4x+10)+3=63; \therefore \tfrac{2}{3}(4x+10)=60;$$

$\therefore 8x+20=180; \therefore 8x=160; \therefore x=20; \therefore 5x=100$, the number of shillings he began with.

[15.] Let $x$ be the number of days, then as $\frac{9}{25}$ths of the stock had been consumed in the nine days, only $\frac{16}{25}$ths of it remained for the 750 men; hence the allowance for one man for one day must have been $\frac{16}{25}\div 750x$; but by the question, the allowance for one man per day was $\frac{1}{25000}$ of the whole; therefore,

$$\frac{1}{25000}=\frac{16}{25\times 750x}; \therefore 25\times 750x=16\times 25000;$$

$\therefore 75x=1600; \therefore 15x=320; \therefore x=21\frac{1}{3}$, the number of days. This question might have been easily solved by the *Rule of Three.*

[16.] Let $x$ represent the number of pounds in *A's* debt, then *B's* was $60-x$, and *C's* $80-x$; consequently *B's* and *C's* together was $140-2x=92$ by the question;

$$\therefore 48=2x; \therefore x=24;$$

hence *A's* debt was £24, *B's* £36, and *C's* £56.

[17.] Let $x$ be the number of sheep, then $\frac{x}{10}$ must be the number of acres ploughed, and $\frac{x}{4}$ the number for pasturage; consequently by the question,

$$\frac{x}{10}+\frac{x}{4}=700;\ \therefore 2x+5x=14000;\ \therefore x=2000.$$

[18.] Suppose he takes $\frac{1}{x}$th of a gallon of the first, then he must take $1-\frac{1}{x}$ of a gallon of the second. The value of the first portion is $\frac{20}{x}$ shillings; and that of the second portion, $12\left(1-\frac{1}{x}\right)$ shillings: but the value of both portions together is, by the question, 14*s.*; therefore

$$\frac{20}{x}+12-\frac{12}{x}=14;\ \therefore \frac{8}{x}=2;\ \therefore 8=2x;\ \therefore x=4;$$

consequently he must take $\frac{1}{4}$ of the first, and ∴ $\frac{3}{4}$ of the second.

[19.] Let $x$ represent the income, then by the question, *A* spends $\frac{4x}{5}$, and *B* spends $\frac{4x}{5}+80$, yearly; consequently in 4 years *B* spends $\frac{16x}{5}+320$; while the income for 4 years is $4x$; hence by the question,

$$\frac{16x}{5}+320=4x+220;\ \therefore \frac{16x}{5}-4x=-100;$$

$$\therefore 16x-20x=-500;\ \therefore x=\frac{500}{4}=125;$$

consequently the income is £125; *A's* expenditure is

$$£\frac{4\times125}{5}=£4\times25=£100,\ \text{and } B\text{'s, } £180.$$

[20.] Let the *first* part be $x$, then since this, increased by 5, is equal to the second diminished by 4, it follows that the *second* must be $x+5+4$; also, since $x+5$ is the third multiplied by 3, the *third* must be $\frac{x+5}{3}$; and the *fourth* must be $2(x+5)$; hence the sum of the four parts is,

$$x+x+9+\frac{x+5}{3}+2x+10=116, \text{ by the question,}$$

that is, $4x+\frac{x+5}{3}=97$; $\therefore 12x+x+5=291$;

$\therefore 13x=286$; $\therefore x=\frac{286}{13}=22$, the first part;

$\therefore x+9=31$, the second part; $\frac{x+5}{3}=9$, the third part; and $2(x+5)=54$, the fourth part.

[21.] Suppose the number of pence he gave to be $x$, then

the father had $\frac{x}{2}+1$, when there was $\frac{x}{2}-1$ left;

the mother had $\frac{1}{2}\left(\frac{x}{2}-1\right)+2$, leaving

$$\frac{x}{2}-1-\frac{1}{2}\left(\frac{x}{2}-1\right)-2=\frac{1}{2}\left(\frac{x}{2}-1\right)-2;$$

and the children had $\frac{1}{4}\left(\frac{x}{2}-1\right)-1+3$;

adding these together, the whole amount is

$$\frac{x}{2}+\frac{3}{4}\left(\frac{x}{2}-1\right)+5=x; \quad \therefore \frac{3}{4}\left(\frac{x}{2}-1\right)+5=\frac{x}{2};$$

multiplying by 4, $\frac{3x}{2}-3+20=2x$;

$\therefore 17=\frac{x}{2}$; $\therefore x=34$, the number of pence;

consequently he gave 2*s*. 10*d*.

*Otherwise*, let $4x$ represent the number of pence, then
the father had $2x+1$; $\therefore 2x-1$ was left;
the mother had $x-\frac{1}{2}+2$, leaving $x-\frac{1}{2}-2$;
and the children had $\frac{1}{2}(x-\frac{1}{2})-1+3$:
the sum of these is $2x+\frac{3}{2}(x-\frac{1}{2})+5=4x$;
$\therefore \frac{3}{2}(x-\frac{1}{2})+5=2x$; $\therefore 3x-\frac{3}{2}+10=4x$;
$\therefore x=10-\frac{3}{2}=\frac{17}{2}$; $\therefore 4x=34$, the number of pence.

[22.] Suppose $B$ and $C$ can finish the work in $x$ days, then as $B$ does twice as much as $C$, $3x$ must be the number of days that $C$ alone could do it in; therefore $\frac{1}{3x}$ is the part of the whole done by $C$ in 1 day, and by the question, $\frac{1}{8}$ of the whole is what is done by $A$ and $C$ together in 1 day; ∴ $\frac{1}{8}-\frac{1}{3x}$ is the part done by $A$ in 1 day; also $\frac{2}{3x}$, being twice the work of $C$, is what $B$ does in 1 day; ∴ $A$ and $B$ together in 1 day do

$$\frac{1}{8}-\frac{1}{3x}+\frac{2}{3x}=\frac{1}{6}, \text{ by the question;}$$

that is,
$$\frac{1}{8}+\frac{1}{3x}=\frac{1}{6};$$

∴ multiplying by $24x$, $3x+8=4x$; ∴ $x=8$; consequently $B$ and $C$ together can do the work in 8 days.

This question may be very readily solved in a different manner; thus, let $A$, $B$, $C$ represent, not the workmen, but what they can accomplish in one day; then, since $B = 2C$, we have by the first condition, for *one* day's work,

$$\begin{array}{rl} & A+2C=\frac{1}{6} \text{ of the whole;} \\ \text{and by the second,} & \underline{A\ +\ C\ =\frac{1}{8}\qquad\quad ,,} \\ & \therefore C\ =\frac{1}{24}\qquad\quad ,, \end{array}$$

and since $B+C=3C$, we have $B+C=\frac{1}{8}$, the part $B$ and $C$ can do in 1 day conjointly; consequently they can do 8 times as much, that is, the whole work, in 8 days; which is the answer.

[23.] Let $x$ be one part, then $a-x$ is the other; and by the question, $x^2+b=(a-x)^2-b=a^2-2ax+x^2-b$;

transposing; $2ax=a^2-2b$; ∴ $x=\dfrac{a^2-2b}{2a}$, one part,

and $a-x=a-\dfrac{a^2-2b}{2a}=\dfrac{a^2+2b}{2a}$, the other part.

[24.] Let $x$ = the number of lb. of inferior sort,
then $224-x$ = . . . . . . . . . . . . . . . . . superior sort.

and as 182 lb. of the former fill the cask, the portion of the cask occupied by $x$ of these lb. must be $\frac{x}{182}$, also the portion occupied by the $224-x$ lb. must be $\frac{224-x}{230}$, and the sum of these portions make up the *whole*, or 1 cask; therefore

$$\frac{x}{182}+\frac{224-x}{230}=1;$$

clearing fractions, $230x+40768-182x=41860$;

transposing, $48x=1092$; $\therefore x=\frac{1092}{48}=22\frac{3}{4}$;

$$\therefore 224-x=224-22\tfrac{3}{4}=201\tfrac{1}{4};$$

hence there must be $22\frac{3}{4}$ lb. of the inferior sort, and $201\frac{1}{4}$ lb. of the superior sort.

It may be remarked here that there is no other way in which a full cask could be made to weigh 2 cwt.; so that the mixture here determined is not only the *best*, it is the *only* mixture that could be sent under the conditions. If $x$, instead of denoting the number of lb. of the inferior, had represented those of the superior, the solution would have been as follows:

$$\frac{x}{230}+\frac{224-x}{182}=1;$$

and clearing fractions,

$$182x+51520-230x=41860;$$

$$\therefore 9660=48x; \therefore x=\tfrac{9660}{48}=201\tfrac{1}{4},$$

the number of lb. of superior sort, as before.

[25.] Imitating the first solution above, we have the equation,

$$\frac{x}{a}+\frac{c-x}{b}=1;$$

clearing, $bx+ac-ax=ab$; $\therefore (b-a)x=a(b-c)$;

$$\therefore x=\frac{a(b-c)}{b-a}, \text{ the number of lb. of inferior;}$$

$$\therefore c-x=\frac{b(c-a)}{b-a}, \text{ . . . . . . . . . . . . . superior.}$$

[26.] Suppose he takes $x$ gallons at $a$ shillings, then he must take $d-x$ gallons at $b$ shillings.

The worth of the former is $ax$ shillings, and that of the latter $b(d-x)$ shillings, and since the worth of the whole mixture is to be $cd$ shillings, we have the equation,

$ax+b(d-x)=cd$, or $ax+bd-bx=cd$;

$$\therefore (a-b)x=(c-b)d;\ \therefore\ x=\frac{c-b}{a-b}d, \text{ gallons of first};$$

$$\therefore d-x=\frac{a-c}{a-b}d, \text{ gallons of second.}$$

[27.] Suppose his original stock was $x$ pounds, then at the end of the first year his unexpended stock was $x-50$; and as this was increased one-third, he must have had

$$\frac{4}{3}(x-50)=\frac{4x-200}{3} \text{ at the end of the first year.}$$

At the end of the second year, the unexpended portion of this was $\frac{4x-200}{3}-50=\frac{4x-350}{3}$, which being increased one-third, gives $\frac{4}{3}\left(\frac{4x-350}{3}\right)$ for the number of pounds at the end of the second year. The unexpended portion of this at the end of the third year was, $\frac{16x-1400}{9}-50=\frac{16x-1850}{9}$, which, being increased one-third, gives $\frac{4}{3}\left(\frac{16x-1850}{9}\right)$ for the number of pounds at the end of the third year, and this number, by the question, is $2x$.

$$\therefore \frac{4}{3}\left(\frac{16x-1850}{9}\right)=2x;$$

or, multiplying by 27 and dividing by 2,

$$32x-3700=27x;\ \therefore 5x=3700;\ \therefore x=740;$$

consequently the original stock was £740.

[28.] Suppose they were engaged for $x$ days, then $A$ worked $x-4$ days, and $B$ $x-7$ days; consequently the number of shillings the former earned per day was $\frac{75}{x-4}$, and the number the latter earned was $\frac{48}{x-7}$. The question states that $x-7$ days, at the former rate of wages, is the same as $x-4$ days at the latter rate; therefore

$$\frac{75}{x-4}(x-7)=\frac{48}{x-7}(x-4), \text{ or } \frac{25}{x-4}(x-7)=\frac{16}{x-7}(x-4);$$

clearing fractions, $25(x-7)^2=16(x-4)^2$;
therefore, taking the square root of each side,

$$5(x-7)=4(x-4), \text{ or } 5x-35=4x-16; \therefore x=19;$$

consequently they were engaged for 19 days.

$A$ worked $x-4$ days, $=15$ days; $B$ worked $x-7$ days, $=12$ days; $A$ received $\frac{75}{15}s. = 5s.$ per day; and $B$ $\frac{48}{12}s. = 4s.$ per day.

[29.] Suppose the first commenced with $x$ pounds, then
at the end of the 1st year he had $2x-100$;
at the end of the 2d year he had $4x-200-100=4x-300$;
at the end of the 3d year he had $8x-600-100=8x-700$;
consequently, by the question, $8x-700=2x$;

$$\therefore 6x=700; \therefore £x=£\tfrac{700}{6}=£116\ 13s.\ 4d.$$

But in the case of the other trader, the equation is

$$8x-700=\frac{x}{2}; \qquad \therefore 16x-1400=x;$$

$$\therefore 15x=1400; \therefore £x=£\tfrac{1400}{15}=£93\ 6s.\ 8d.;$$

consequently one commenced with £116 13*s.* 4*d.*, and the other with £93 6*s.* 8*d.*

[30.] Suppose the work has been in hand $x$ days, then as the first two labourers can do $\frac{1}{4}$ and $\frac{1}{7}$ respectively of the whole work in a day, in $x$ days they do together $\frac{x}{4}+\frac{x}{7}$; also, since the third labourer can do $\frac{1}{10}$ in a day, he does $\frac{x-1}{10}$ in the $x-1$ days he remains. The question states that the sum of all these portions of work is $\frac{4}{5}$ of the whole: therefore,

$$\frac{x}{4}+\frac{x}{7}+\frac{x-1}{10}=\frac{4}{5};$$

or, multiplying by 10, $\frac{5x}{2}+\frac{10x}{7}+x-1=8$;

that is, $\frac{7x}{2}+\frac{10x}{7}=9$; $\therefore 49x+20x=126$;

$\therefore 69x=126$; $\therefore x=\frac{126}{69}=1\frac{19}{23}$;

consequently the work had been in hand $1\frac{19}{23}$ days.

[31.] Suppose the hands are together at $x$ minutes past 1 o'clock; then the long hand must have passed over 12 five-minute spaces, together with these $x$ minutes; that is, it must have passed over $60+x$ minute spaces. But it moves twelve times as fast as the short hand;

therefore $12x=60+x$; $\therefore 11x=60$; $\therefore x=\frac{60}{11}=5\frac{5}{11}$, the number of minutes past 1.

[32.] When the long hand is at XII, the short hand is at V; the long hand then moves down and overtakes the short hand; suppose it comes up to it at $x$ minutes past the V, then, while the long hand has moved over $5+x$ five-minute spaces, the short hand has moved over $x$ minutes: but the former moves 12 times as fast as the latter;

$\therefore 12x=25+x$; $\therefore 11x=25$; $\therefore x=\frac{25}{11}=2\frac{3}{11}$;

consequently the hands are together at $2\frac{3}{11}$ minutes past the V, that is, at $27\frac{3}{11}$ past five o'clock.

[33.] Suppose they finish it in $x$ days; then they can do $\frac{1}{x}$ of the whole in 1 day; but $A$ and $B$ do $\frac{1}{3}$ of the whole in 1 day; $\therefore \frac{1}{x}-\frac{1}{3}$ is the part $C$ alone can do in 1 day. Again, $A$ and $C$ do $\frac{1}{5}$ in 1 day; $\therefore \frac{1}{x}-\frac{1}{5}$ is the part $B$ alone can do in 1 day; and in like manner $\frac{1}{x}-\frac{1}{6}$ is the part *A* alone can do in 1 day: hence the sum of these parts is what they can all three do conjointly in 1 day, that is, it is $\frac{1}{x}$;

$$\therefore \frac{1}{x}-\frac{1}{3}+\frac{1}{x}-\frac{1}{5}+\frac{1}{x}-\frac{1}{6}=\frac{1}{x};$$

$$\text{or } \frac{2}{x}-\frac{1}{3}-\frac{1}{5}-\frac{1}{6}=0; \ \therefore \frac{2}{x}=\frac{21}{30}; \ \therefore \frac{1}{x}=\frac{21}{60};$$

$$\therefore x=\tfrac{60}{21}=2\tfrac{6}{7};$$

consequently they all three finish the work in $2\frac{6}{7}$ days.

This question can be easily worked as follows: let *A*, *B*, *C*, represent what these agents can respectively accomplish in 1 day; then by the question,

$$\begin{aligned} A+\ B&=\tfrac{1}{3}\\ A+\ C&=\tfrac{1}{5}\\ B+\ C&=\tfrac{1}{6}\\ \hline 2A+2B+2C&=\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{6}=\tfrac{21}{30};\end{aligned}$$

$$\therefore A+B+C=\tfrac{21}{60}, \text{ the part done in 1 day;}$$

consequently there will be as many days occupied in doing the whole as there are such parts in one whole; that is, the number of days will be $1\div\frac{21}{60}=\frac{60}{21}=2\frac{6}{7}$, as before.

[34.] Proceeding as in last example, we have,

$$\begin{aligned}\text{day's work of } C &= \frac{1}{x}-\frac{1}{a};\\ \text{ditto} \quad \text{of } B &= \frac{1}{x}-\frac{1}{b};\\ \text{ditto} \quad \text{of } A &= \frac{1}{x}-\frac{1}{c};\end{aligned}$$

$$\therefore \frac{3}{x}-\frac{ab+ac+bc}{abc}=\frac{1}{x}; \quad \therefore \frac{2}{x}=\frac{ab+ac+bc}{abc};$$

$$\therefore x=\frac{2abc}{ab+ac+bc}, \text{ the number of days.}$$

*Otherwise,*
$$\left.\begin{aligned} A+B&=\frac{1}{a}\\ A+C&=\frac{1}{b}\\ B+C&=\frac{1}{c}\end{aligned}\right\}\text{ work done in 1 day;}$$

$$2A+2B+2C=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+ac+bc}{abc};$$

$\therefore A+B+C=\dfrac{ab+ac+bc}{2abc}$, the part done in 1 day;

and $\therefore$ 1, divided by this, that is, $\dfrac{2abc}{ab+ac+bc}=$ the number of days.

[35.] By the first solution above, the parts of the whole done by $A$, $B$, $C$, respectively, in 1 day are,

$$\frac{1}{x}-\frac{1}{c};\ \frac{1}{x}-\frac{1}{b};\ \frac{1}{x}-\frac{1}{a};\text{ where }\frac{1}{x}=\frac{ab+ac+bc}{2abc};$$

hence, by substituting this value, the parts are,

$$\frac{ac+bc-ab}{2abc};\ \frac{ab+bc-ac}{2abc};\ \frac{ab+ac-bc}{2abc};$$

so that, dividing 1 by each of these, that is, taking their reciprocals, we have,

$$\frac{2abc}{ac+bc-ab},\ \frac{2abc}{ab+bc-ac},\ \frac{2abc}{ab+ac-bc},$$

for the required number of days.

[36.] Suppose the man alone could empty the cask in $x$ days, then his share for 1 day is $\frac{1}{x}$; and as $\frac{1}{15}$ is the part of the whole which both together consume in 1 day, it follows that $\frac{1}{15}-\frac{1}{x}$ is the woman's daily share; also for 6 days the joint consumption is $\frac{6}{15}$; and by the question, the remainder, namely $1-\frac{6}{15}$, or $\frac{3}{5}$, is what the woman consumes in 30 days: but in 30 days she drinks

$$\frac{30}{15}-\frac{30}{x}, \text{ or } \frac{10}{5}-\frac{30}{x};$$

$$\therefore \frac{3}{5}=\frac{10}{5}-\frac{30}{x}; \quad \therefore \frac{30}{x}=\frac{7}{5}; \quad \therefore 150=7x;$$

$\therefore x=\frac{150}{7}=21\frac{3}{7}$, the number of days the man takes; and as the woman drinks $\frac{3}{5}$ in 30 days, she drinks $\frac{1}{5}$ in 10 days, or the whole in 50 days.

[37.] Let $x$ be the number of leaps taken by the fox after starting, then the greyhound must pass over $x+60$ of the fox's leaps before catching him, and this he must do in the time it takes the fox to make $x$ leaps. Now by the question, while the fox makes $x$ leaps, the greyhound makes $\frac{6}{9}x$, or $\frac{2}{3}x$ leaps; but these are $\frac{7}{3}$ times as great as the fox's leaps;

$\therefore \frac{14}{9}x=x+60; \quad \therefore 14x=9x+540;$

$\therefore 5x=540; \quad \therefore x=108$, the number of fox's leaps;

$\therefore \frac{2}{3}x=72$, the number of greyhound's.

*Otherwise.* Let $x$ be the number of leaps taken by the greyhound, then the number of the fox's leaps, *plus* 60, is $\frac{3}{2}x+60$, which is the whole number of fox's leaps in the distance between the greyhound's starting point and the point of capture, and which distance is also $x$ of the greyhound's leaps: but 3 of these are in length equal to 7 of the other;

$$\therefore 7x=\frac{9x}{2}+180; \quad \therefore 14x=9x+360;$$

$$\therefore 5x=360; \quad \therefore x=72.$$

[38.] Let $x$ be the first number, then since, if divisor be multiplied by quotient, and remainder taken in, the result will be the dividend: we have by the conditions of the question the following equations, namely,

$$\begin{aligned} x &= \text{the first number,} \\ 2x+1 &= \text{the second,} \\ \underline{6x+3+3} &= \text{the third;} \\ \therefore 9x+7 &= 70, \text{ by the question;} \end{aligned}$$

$\therefore 9x=63; \quad \therefore x=7; \quad 2x+1=15; \quad 6x+6=48;$

consequently the three numbers are 7, 15, and 48.

[39.] Let the time be $x$ of the units expressed in the times $d, e, f$, that is, hours or days, &c.; then, in 1 of these units of time, $\frac{1}{x}$ of the effect $E$ can be produced, that is, the part $\frac{E}{x}$. But the joint effect produced by the agents in a unit of time is, by the question, $\frac{a}{d}+\frac{b}{e}+\frac{c}{f}=\frac{aef+bdf+cde}{def}$;

$$\therefore \frac{aef+bdf+cde}{def}=\frac{E}{x};$$

$$\therefore x=\frac{def}{aef+bdf+cde}E,$$

the number of units of time.

[40.] From the solution to example 31, it appears that the hands are again together after a lapse of 1 h $5\frac{5}{11}$ min.; and as they move uniformly, it follows that *whenever* they are together, they must always be together again after the lapse of this interval of time; so that in the 12 hours after their first start, they are together as often as 1 h. $5\frac{5}{11}$ min. is contained in 12 h., that is to say 11 times; the several conjunctions being at $5\frac{5}{11}$ min. past 1; at $10\frac{10}{11}$ min. past 2; at $16\frac{4}{11}$ min. past 3, and so on, each conjunction being 1 h. $5\frac{5}{11}$ min. after the immediately preceding one.

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## SIMULTANEOUS SIMPLE EQUATIONS.—PAGE 135.

[1.] $\left.\begin{aligned}4x-3y&=\ \ 7\ \ldots (1)\\ 5x+2y&=14\ \ldots (2)\end{aligned}\right\}$

add, $9x-\ \ y=21\ \ldots (3)$;

$\therefore 18x-2y=42$;

adding this to (2),

$23x=56$; $\therefore x=\frac{56}{23}$;

$\therefore$ (3), $y=9x-21=\frac{21}{23}$.

[2.] $\left.\begin{aligned}6x+\ \ y&=36\ \ldots (1)\\ 8x+5y&=26\ \ldots (2)\end{aligned}\right\}$

sub., $2x+4y=-10$

(1)×4, $24x+4y=144$;

sub., $22x=154$; $\therefore x=7$;

$\therefore$ (1), $y=36-6x=-6$.

[3.] $13x - 3y = 65 \ldots (1)$
$9x + 2y = 98 \ldots (2);$
add, $22x - y = 163 \ldots (3);$
$\therefore 44x - 2y = 326;$
adding this to (2),
$53x = 424; \therefore x = 8;$
$\therefore$ (3), $y = 22x - 163 = 13.$

[4.] $x + 49y = 693$
$49x + y = 357;$
adding and subtracting,
$50x + 50y = 1050$
$48y - 48x = 336;$
$\therefore x + y = \frac{105}{5} = 21$
$y - x = \frac{336}{48} = 7;$
adding and subtracting,
$2y = 28; \therefore y = 14$
$2x = 14; \therefore x = 7.$

[5.] $3x + 2y = 42$
$2x + 3y = 48;$
sub, $-x + y = 6 \ldots (1);$
add., $5x + 5y = 90;$
$\therefore x + y = 18 \ldots (2);$
adding and sub. (1) and (2),
$2y = 24; \therefore y = 12$
$2x = 12; \therefore y = 6.$

[6.] $x - 6y = 6 \ldots (1)$
$2x - 6y = 15;$
sub., $x = 9;$
$\therefore$ (1), $y = \frac{x-6}{6} = \frac{3}{6} = \frac{1}{2}.$

[7.] $12x - y = 28, (1); \therefore y = 12x - 28$
$4y - x = 29, (2);$
that is, $4(12x - 28) - x = 29;$
or, $47x - 112 = 29;$
$\therefore 47x = 141; \therefore x = 3;$
$\therefore \quad y = 12x - 28 = 8.$

[8.] $6x - 5y = 39 \ldots (1)$
$7x - 3y = 54;$
sub. $x + 2y = 15 \ldots (2);$
$\therefore 6x + 12y = 90;$
sub. (1), $17y = 51; \therefore y = 3;$
$\therefore$ (2), $x = 15 - 2y = 9.$

[9.] $2x - 3y = 17 \ldots (1)$
$3x + 2y = 0 \ldots (2).$
Multiplying (1) by 2, and (2) by 3,

$$\begin{array}{rl} 4x - 6y = & 34 \\ \underline{9x + 6y} = & 0 \\ \text{Sum, } \overline{13x \quad} = & \overline{34}; \quad \therefore x = \frac{34}{13}; \end{array}$$

$\therefore$ (2), $y = -\frac{3x}{2} = -\frac{51}{13}.$

[10.] $ax + by = 1 \ldots (1)$

$bx - cy = 1 \ldots (2);$

multiplying (1) by $b$, and (2) by $a$,

$$abx + b^2y = b$$
$$\underline{abx - acy = a}$$

Diff., $(ac + b^2)y = b - a$; $\therefore y = \dfrac{b-a}{ac+b^2}$;

$$\therefore (1),\ x = \frac{1-by}{a} = \frac{ac+b^2-b^2+ab}{a(ac+b^2)} = \frac{b+c}{ac+b^2}.$$

[11.] $3x + 2y = 102 \ldots (1)$

$4x - \ \ y = \ \ 48 \ldots (2);$

multiplying (2) by 2,

$8x - 2y = 96;$

adding this to (1),

$11x = 198$; $\therefore x = 18$;

$\therefore (2),\ y = 4x - 48 = 24.$

[12.] $x - 15y = \ \ 6 \ldots (1)$

$3x - \ \ 7y = 56 \ldots (2);$

multiplying (1) by 3,

$3x - 45y = 18;$

subtracting this from (2),

$38y = 38$; $\because y = 1$;

$\therefore (1),\ x = 15y + 6 = 21.$

[13.] $3x + 2y = \ \ 48 \ldots (1)$

$5x - 3y = -15 \ldots (2);$

multiplying (1) by 3, and (2) by 2,

$$9x + 6y = \ \ 144$$
$$\underline{10x - 6y = -30}$$

Sum, $19x = 114$; $\therefore x = 6$;

$$\therefore (1),\ y = \frac{48-3x}{2} = 15.$$

[14.] $\dfrac{1}{x} + \dfrac{1}{y} = 12 \ldots (1)$

$$-\frac{1}{y} = 1;\ \ \therefore y = -1;$$

$$\therefore (1),\ \frac{1}{x} = 13;\ \therefore x = \frac{1}{13}.$$

[15.] $ax' + by' = m \ldots (1)$ (Alg. p. 137,

$bx' + ay' = n \ldots (2);$ foot-note.)

Multiplying (1) by $b$, and (2) by $a$,

$$abx' + b^2y' = bm$$
$$\underline{abx' + a^2y' = an}$$

Diff., $(a^2 - b^2)y' = an - bm$

$$\therefore y' = \frac{an-bm}{a^2-b^2}.$$

Again, mult. (1) by $a$, and (2) by $b$,

$$\begin{aligned} a^2x' + aby' &= am \\ b^2x' + aby' &= bn \\ \text{Diff., } (a^2-b^2)x' &= am-bn; \\ \therefore x' &= \frac{am-bn}{a^2-b^2}; \end{aligned}$$

consequently $x = \dfrac{1}{x'} = \dfrac{a^2-b^2}{am-bn}$;

and $y = \dfrac{1}{y'} = \dfrac{a^2-b^2}{an-bm}$.

[16.]

$$\begin{aligned} 21x + 5y &= 297 \\ 5x + 21y &= 249 \\ \text{Sum, } 26x + 26y &= 546; \\ \text{Diff., } 16x - 16y &= 48; \\ \therefore x + y &= \tfrac{546}{26} = 21 \\ x - y &= \tfrac{48}{16} = 3 \\ \text{Sum, } 2x &= 24 \\ \text{Diff., } 2y &= 18; \\ \therefore x = 12,\ y &= 9. \end{aligned}$$

[17.]

$$\begin{aligned} 3(3x-2y) &= 2(2x-7) \\ 4(x+3y) &= 12-3x; \\ \therefore 5x - 6y &= -14 \ldots (1). \\ 7x + 12y &= 12 \ldots (2); \end{aligned}$$

multiplying (1) by 2, $10x - 12y = -28$;
adding this to (2), $17x = -16$; $\therefore x = -\frac{16}{17}$;

$$\therefore (1),\ y = \frac{5x+14}{6} = \frac{14.17-5.16}{6.17} = \frac{14.16+14-5.16}{6.17} = \frac{9.16+14}{6.17} = \frac{9.8+7}{3.17} = \frac{79}{51}.$$

[18.]

$$\begin{aligned} 4(2y-x)-3y+5 &= 0 \\ 3y-5-2y+x &= 3; \\ \therefore 5y-4x &= -5 \ldots (1) \\ y + x &= 8 \ldots (2) \\ \therefore y &= 8-x \ldots (3); \\ \therefore (1),\ 5(8-x)-4x &= -5; \\ \therefore -9x = -45;\ \therefore x &= 5; \\ \therefore (3),\ y = 8-5 &= 3. \end{aligned}$$

[19.]

$$\begin{aligned} 4(x+y)+3(x-y) &= 708 \\ 5x &= 33y; \\ \therefore 7x+y &= 708 \ldots (1) \\ 5x-33y &= 0 \ldots (2). \\ \text{From (1), } y &= 708-7x \ldots (3); \\ \therefore (2),\ 5x-33(708-7x) &= 0; \\ \therefore 5x+231x &= 33.708; \\ \therefore 236x &= 33.708; \\ \therefore x = \frac{33.177}{59} &= 99; \\ \therefore (3),\ y = 708-693 &= 15. \end{aligned}$$

[20.]

$$\begin{aligned} 2x + {\cdot}4y &= 1{\cdot}2 \ldots (1) \\ 3{\cdot}4x - {\cdot}02y &= {\cdot}01 \ldots (2). \end{aligned}$$

Multiplying (2) by 20, $68x - {\cdot}4y = {\cdot}2$.

Adding this to (1), $70x = 1{\cdot}4$; $\therefore x = \dfrac{{\cdot}14}{7} = {\cdot}02$;

$$\therefore (1),\ y = \frac{1{\cdot}2 - {\cdot}04}{{\cdot}4} = \frac{1{\cdot}16}{{\cdot}4} = 2{\cdot}9.$$

[21.] $x + y = 19 \ldots (1)$

$$x^2 - y^2 = 95 \ldots (2);$$

dividing (2) by (1),

$$x - y = 5;$$

adding this to (1), and then subtracting,

$$2x = 24; \therefore x = 12$$
$$2y = 14; \therefore y = 7.$$

[22.] $x + y = a \ldots (1)$

$$x^2 - y^2 = b \ldots (2);$$

dividing (2) by (1)

$$x - y = \frac{b}{a};$$

adding this to (1), and then subtracting,

$$2x = a + \frac{b}{a}; \therefore x = \tfrac{1}{2}\left(a + \frac{b}{a}\right)$$
$$2y = a - \frac{b}{a}; \therefore y = \tfrac{1}{2}\left(a - \frac{b}{a}\right).$$

[23.] $15x - 25y + 30 = 4x + 2y$

$$96 - 3x + 6y = 6x + 4y;$$
$$\therefore 11x - 27y = -30 \ldots (1)$$
$$-9x + 2y = -96 \ldots (2);$$

Multiplying (1) by 9, and (2) by 11,

$$99x - 243y = -270$$
$$-99x + 22y = -1056$$

Sum, $-221y = -1326;$

$$\therefore y = \frac{1326}{221} = 6; \therefore (2),\ x = \frac{2y + 96}{9} = 12.$$

[24.] $a(x - y) = b(x + y) \ldots (1)$

$$x^2 - y^2 = c \ldots (2);$$

multiplying (1) by $x + y$, and (2) by $a$,

$$a(x^2 - y^2) = b(x + y)^2$$
$$a(x^2 - y^2) = ac;$$
$$\therefore (x + y)^2 = \frac{ac}{b}; \therefore x + y = \sqrt{\frac{ac}{b}};$$
$$\therefore (1),\ x - y = \frac{b}{a}\sqrt{\frac{ac}{b}};$$

∴ adding and subtracting,

$$2x = \left(1 + \frac{b}{a}\right)\sqrt{\frac{ac}{b}} = \frac{a + b}{ab}\sqrt{abc}$$
$$2y = \left(1 - \frac{b}{a}\right)\sqrt{\frac{ac}{b}} = \frac{a - b}{ab}\sqrt{abc};$$
$$\therefore x = \frac{a + b}{2ab}\sqrt{abc},\ y = \frac{a - b}{2ab}\sqrt{abc}.$$

[25.] $$\frac{\sqrt{(4x+1)}+\sqrt{y}}{\sqrt{(4x+1)}-\sqrt{y}}=9\ldots(1)$$
$$9x-5y=5\ldots(2);$$

applying the principle at p. 113, Algebra, to (1),

$$\sqrt{\frac{4x+1}{y}}=\frac{5}{4};$$

$$\text{squaring, } \frac{4x+1}{y}=\frac{25}{16};$$

$$\therefore 64x-25y=-16;$$

$$\begin{aligned}(2)\times 5,\ \ 45x-25y &= \phantom{-}25\\ \text{Diff.}\qquad 19x &= -41;\end{aligned}$$

$$\therefore x=-\frac{41}{19};\quad \therefore (2),\ y=\frac{9x}{5}-1=-4\frac{84}{95}.$$

[26.] $$\left.\begin{aligned}\frac{\sqrt{x}+6}{\sqrt{x}+4}&=\frac{\sqrt{y}+2}{\sqrt{y}-2}\\ 2\sqrt{x}+3\sqrt{y}&=2\end{aligned}\right\};$$ applying the principle at page 113 of the Algebra to the first of these, we have,

$$\sqrt{x}+5=\frac{\sqrt{y}}{2};\ \therefore 2\sqrt{x}-\sqrt{y}=-10;$$

and subtracting this from the second equation, we have,

$$4\sqrt{y}=12;\ \therefore \sqrt{y}=3;\ \therefore y=3^2;$$

$$\therefore \sqrt{x}=\frac{\sqrt{y}}{2}-5=-\tfrac{7}{2};\ \therefore x=(-\tfrac{7}{2})^2.$$

The value of $y$ is expressed in the form $y=3^2$, instead of in the form $y=9$, because the step from which this value is deduced shows that $\sqrt{y}$ is exclusively *plus* 3, and not ambiguously $\pm 3$. The square root of 9 is either 3, or $-3$; but the square root of $3^2$ is exclusively 3; and the condition above shows that this restriction is necessary.

In like manner $\sqrt{x}=\frac{3}{2}-5=-\frac{7}{2}$; so that $x$ is strictly $(-\frac{7}{2})^2$. By substituting the values of $x$ and $y$ in these forms, we indicate the proper sign which the square root of each is to take, in order that the proposed equations may be verified upon substituting these roots for $\sqrt{x}$ and $\sqrt{y}$. (See the remarks at page 68 of this Key.)

[27.]
$$ax+by=c;\;.\;.\;.\;.\;.\;.\;.\;(1)$$
$$a^2+ax=b^2+by;$$
$$\therefore ax-by=b^2-a^2.\;.\;.\;.\;(2)$$

Adding and subtracting (1) and (2),

$$2ax=c-a^2+b^2;\;\therefore x=\frac{c-a^2+b^2}{2a}$$
$$2by=c+a^2-b^2;\;\therefore y=\frac{c+a^2-b^2}{2b}.$$

[28.]
$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cx}=1$$
$$\frac{1}{ay}+\frac{1}{bx}-\frac{1}{cy}=1;$$

that is, putting $x'$, $y'$, $z'$, for the reciprocals of $x$, $y$, $z$, and clearing fractions,

$$(a+c)bx'+acy'=abc\;.\;.\;.\;(1)$$
$$acx'-(a-c)by'=abc\;.\;.\;.\;(2).$$

Equalizing the coefficients of $y'$,

$$(a^2-c^2)b^2x'+(a-c)abcy'=(ab-bc)abc$$
$$a^2c^2x'-(a-c)abcy'=(ac)abc$$
$$\text{Sum, }(a^2b^2+a^2c^2-b^2c^2)x'=(ab+ac-bc)abc;$$
$$\therefore x'=\frac{(ab+ac-bc)abc}{a^2(b^2+c^2)-b^2c^2}.$$

In like manner, equalizing the coefficients of $x'$ in (1), (2),

$$(a+c)abcx'+a^2c^2y'=ac(abc)$$
$$(a+c)abcx'-(a^2-c^2)b^2y'=(ab+bc)abc.$$
$$\text{Diff. }(a^2b^2+a^2c^2-b^2c^2)y'=(ac-ab-bc)abc;$$
$$\therefore y'=\frac{(ac-ab-bc)abc}{a^2(b^2+c^2)-b^2c^2};$$

and the reciprocals of these expressions for $x'$ and $y'$ are the values of $x$ and $y$.

[29.] The first of these equations is, $\dfrac{4y-3x}{23-3x}=\dfrac{19}{2}$;

and the second is, $\dfrac{2y-3}{3(x-6)}=\dfrac{17}{3}$;

therefore, clearing fractions they become, after transposing,

$$\begin{array}{r}51x+8y=437\ \ldots\ (1)\\ 17x-2y=\ \ 99\ \ldots\ (2);\end{array}$$

$\therefore$ (2) $\times 3 = 51x - 6y = 297$;

subtracting this from (1), $14y = 140$; $\therefore y = 10$.

Again, (2) $\times 4 = 68x - 8y = 396$;

adding this to (1), $119x = 833$; $\therefore x = 7$.

[30.] The first equation is,

$$5\tfrac{1}{3} + \frac{x}{5} = 18\tfrac{1}{3} - \frac{4x+3y-8}{7}, \text{ or } \frac{x}{5} = 13 - \frac{4x+3y-8}{7};$$

and the second is,

$$10y + \frac{6x}{5} - 7 = 55 + 10x, \text{ or } 10y + \frac{x}{5} = 62 + 9x;$$

so that, clearing fractions and transposing, we have,

$$\begin{array}{lll}27x+15y=\ \ \ 495 & \text{or,} & 9x+\ \ 5y=\ \ \ 165\ \ldots\ (1)\\ 44x-50y=-310; & & 22x-25y=-155\ \ldots\ (2)\end{array}$$

$$\begin{array}{rl}(1)\times 5 = & \underline{45x+25y}=825\\ \text{sum,} & \ \ 67x\ \ \ \ \ \ =670;\ \therefore x=10;\end{array}$$

$$\therefore (1),\ y = \frac{165-9x}{5} = 15.$$

## SIMPLE EQUATIONS WITH THREE UNKNOWN QUANTITIES.

### PAGE 139.

[1.]
$$\begin{array}{rl} & x+y+z=31\ \ldots\ (1)\\ & \underline{x+y-z}=25\ \ldots\ (2)\\ \text{diff.} & \ \ \ \ \ \ 2z=\ \ 6;\ \therefore z=3.\end{array}$$

$$\begin{array}{rl}\text{Also,} & x+y+z=31\\ & \underline{x-y-z}=\ \ 9\ \ldots\ (3)\\ \text{sum,} & \ \ 2x\ \ \ \ \ \ =40;\ \therefore x=20;\end{array}$$

$$\therefore (3),\ y = x - z - 9 = 8.$$

[2.] Multiplying the first equation by 2,

$$\begin{array}{rl} & 4x+8y-6z=44\\ & \underline{4x-2y+5z=18}\\ \text{diff.} & 10y-11z=26\ \ldots\ (4).\end{array}$$

Also, multiplying the same first equation by 3,

$$\begin{array}{rl} & 6x+12y-9z=66 \\ & 6x+\ \ 7y-\ \ z=63 \\ \hline \text{diff.} & \quad 5y-8z=\ \ 3\ \ldots (B) \end{array}$$

$$(B)\times 2=10y-16z=\ \ 6;$$

subtracting this from $(A)$, $5z=20$; $\therefore z=4$;

$$\therefore (B),\ y=\frac{3+8z}{5}=7;$$

and from the second of the proposed equations,

$$x=\frac{18+2y-5z}{4}=3.$$

[3.] Multiplying the second equation by 4,

$$\begin{array}{rl} & 3x-\ \ 9y+8z=\ \ \ 41 \\ & -20x+16y+8z=-80 \\ \hline \text{diff.} & \quad 23x-25y=121\ \ldots\ldots (A). \end{array}$$

Also, multiplying the same by 3,

$$\begin{array}{rl} & -15x+12y+6z=-60 \\ & 11x-\ \ 7y-6z=\ \ \ 37 \\ \hline \text{sum,} & -4x+\ \ 5y\qquad =-23\ \ldots (B); \end{array}$$

$$\therefore (B)\times 5=-20x+25y=-115;$$

adding this to $(A)$, $3x=6$; $\therefore x=2$;

$$\therefore (B),\ y=\frac{4x-23}{5}=-3;$$

and from the first of the proposed equations,

$$z=\frac{15x-12y-60}{6}=1.$$

[4.] From the second equation,

$$z=2y-\ \ 11\ \ldots (1);$$

substituting this in the third, we have the two equations,

$$\begin{array}{l} 5x+3y=\ \ 65\ \ldots\ldots\ldots (2) \\ 3x+8y=101\ \ldots\ldots\ldots (3); \end{array}$$

equalizing the coefficients of $x$,

$$\begin{array}{rl} & 15x+\ \ 9y=195 \\ & 15x+40y=505 \\ \hline \text{diff.} & \qquad 31y=310;\ \therefore y=10; \end{array}$$

$$\therefore (1),\ z=9;\ \text{and } (2),\ x=\frac{65-3y}{5}=\ 7.$$

[5.] Clearing fractions, the equations are,

$$\begin{aligned} 5x+2y+\phantom{10}z&=\phantom{1}30\\ 5x+8y-\phantom{10}z&=\phantom{1}40\\ 2x+5y+10z&=129; \end{aligned}$$

adding the first to the second,

$$10x+10y=70;$$
$$\text{or,}\quad x+y=\ 7\ \ldots (A);$$

subtracting the third from 10 times the first,

$$48x+15y=171;$$
$$\text{or,}\quad 16x+\ 5y=\ 57\ \ldots (B);$$
$$\text{from } (A),\quad y=7-x;$$

substituting this in $(B)$,

$$16x-5x=22;\ \therefore x=2, \text{ and } y=7-x=5;$$

and from the first equation, $z=30-5x-2y=1$.

[6.] Clearing fractions, the equations are,

$$\begin{aligned} 6x+\phantom{1}3y+\phantom{1}2z&=192,\\ 20x+15y+12z&=900,\\ 15x+12y+10z&=720; \end{aligned}$$

multiplying the first by 5,

$$30x+15y+10z=960;$$

and subtracting the second,

$$10x-2z=60;$$
$$\text{or,}\quad 5x-\ z=30\ \ldots (A);$$

multiplying the first by 4,

$$24x+12y+8z=768;$$

and subtracting the third, $9x-2z=48\ \ldots (B)$;

$$(A)\times 2=\underline{10x-2z=60}$$
$$\text{diff.}\quad x\quad =12;\ \therefore (A),\ z=30,$$
$$\text{and } y=\frac{192-6x-2z}{3}=20.$$

[7.] From the first equation,

$$y=c-ax\ \ldots\ldots\ldots (A);$$

substituting this in the second,

$$bc-abx+z=a;$$
$$\therefore z=abx-bc+a\ \ldots (B);$$

substituting this in the third,

$$abcx - bc^2 + ac + x = b;$$

$$\therefore x = \frac{b + bc^2 - ac}{abc + 1};$$

$$\therefore (A),\ y = \frac{c + a^2c - ab}{abc + 1};$$

and from the second of the proposed equations,

$$z = a - by = \frac{a + ab^2 - bc}{abc + 1}.$$

[8.] From the sum of the first and second equations subtract the third, and we have

$$\frac{2a}{x} = 1; \quad \therefore x = 2a.$$

From the sum of the first and third subtract the second, and we have

$$\frac{2b}{y} = 1; \quad \therefore y = 2b.$$

From the sum of the second and third subtract the first, and we have

$$\frac{2c}{z} = 1; \quad \therefore z = 2c.$$

[9.] These equations, cleared of fractions, are

$$\begin{aligned} x + y + 6z &= \phantom{-}63 \\ y + z - 6x &= -130 \\ 3x + y - z &= \phantom{-}76; \end{aligned}$$

adding the second and third,

$$-3x + 2y = -54 \ldots (A);$$

also, adding 6 times the third to the first,

$$19x + 7y = 519 \ldots (B);$$

equalizing the coefficients of $y$,

$$\begin{aligned} 21x - 14y &= 378 \\ 38x + 14y &= 1038 \\ \text{Sum, } 59x \phantom{+14y} &= 1416; \ \therefore x = \frac{1416}{59} = 24; \end{aligned}$$

$$\therefore (A),\ y = \frac{3x}{2} - 27 = 9; \text{ and } z = 3x + y - 76 = 5.$$

[10.] These equations are the same as

$$\frac{2x}{5}+\frac{3y+z}{10}-\frac{2y+2z-x+1}{15}=4+\frac{x-z}{5}$$

$$\frac{3x}{4}+\frac{5y}{12}-\frac{z}{6}-\frac{x}{2}-\frac{y-3z}{4}=\frac{7y+z+3}{11}+\frac{1}{6}$$

$$\frac{5y+3z}{4}-\frac{x}{6}-\frac{y}{4}+\frac{z}{12}+2z=y+\frac{x}{2}+\frac{y}{3}+\frac{1}{6};$$

or, $$\frac{x+z}{5}+\frac{3y+z}{10}-\frac{2y+2z-x+1}{15}=4$$

$$\frac{x+3z}{4}+\frac{y-z}{6}=\frac{7y+z+3}{11}+\frac{1}{6}$$

$$\frac{3z}{4}-\frac{x}{6}+\frac{z}{12}+2z=\frac{x}{2}+\frac{y}{3}+\frac{1}{6};$$

clearing fractions,

$$6x+6z+9y+3z-4y-4z+2x-2=120$$
$$33x+99z+22y-22z=84y+12z+36+22$$
$$9z-2x+z+24z=6x+4y+2;$$

transposing,

$$\begin{aligned}8x+5y+5z&=122\ldots(1)\\33x-62y+65z&=58\ldots(2)\\8x+4y-34z&=-2\ldots(3);\end{aligned}$$

subtracting (3) from (1),

$$y+39z=124;\ \therefore y=124-39z\ldots(A);$$

subtracting 4 times (1) from (2),

$$x-82y+45z=-430\ldots\ldots(B);$$

subtracting 8 times this from (3),

$$660y-394z=3438;$$

or, $330y-197z=1719;$

substituting ($A$) in this,

$$40920-13067z=1719;$$

$$\therefore z=\tfrac{39201}{13067}=3;\ \therefore (A),\ y=7;\ \therefore (B),\ x=9.$$

## QUESTIONS IN SIMULTANEOUS EQUATIONS.—PAGE 142.

[1.] Let the price per dozen of the port be $x$ pounds, and that of the sherry $y$ pounds; then, by the question,

$$\begin{aligned} 20x+30y&=120 \qquad \text{or, } 2x+3y=12\ \ldots\ (1)\\ 30x+25y&=140; \qquad\quad\ 6x+5y=28\ \ldots\ (2);\\ \text{multiplying (1) by 3}\ \ldots\ &\quad 6x+9y=36\\ &\text{diff. }\ 4y=8;\ \therefore y=2;\\ &\therefore (1),\ x=\frac{12-3y}{2}=3; \end{aligned}$$

hence the price of the port per dozen was £3, and that of the sherry £2.

[2.] Let the fraction be $\frac{x}{y}$; then, by the question,

$$\begin{aligned} \frac{x+4}{y}&=\frac{1}{2};\ \therefore 2x+8=y\ \ldots\ldots\ (1)\\ \frac{x}{y+7}&=\frac{1}{5};\qquad \therefore 5x=y+7\ \ldots\ (2)\\ &\text{diff. } 3x-8=7;\ \therefore x=\tfrac{15}{3}=5;\\ &\therefore (1),\ y=2x+8=18; \end{aligned}$$

consequently the fraction is $\frac{5}{18}$.

[3.] Let $x$ and $y$ be the digits; then the number must be $10x+y$; and this number inverted is $10y+x$. By the question, $10x+y=4x+4y$

$$\begin{aligned} 10x+y+18&=10y+x;\\ \therefore 6x-3y&=0; \qquad \text{or, } 2x-y=0\ \ldots\ (1)\\ 9x-9y&=-18;\ \text{or, } x-y=-2\ \ldots\ (2)\\ &\text{diff. }\quad x\ =\ 2;\ \therefore (2),\ y=4; \end{aligned}$$

consequently the number is 24.

[4.] Let $\frac{x}{y}$ be the fraction; then, by the question,

$$\begin{aligned} \frac{x+1}{y-1}&=\frac{4}{5};\ \therefore 5x+5=4y-4\ \ldots\ (1)\\ \frac{x-1}{y+1}&=\frac{1}{2};\ \therefore 2x-2=\ y+1\ \ldots\ (2); \end{aligned}$$

multiplying (2) by 4, $8x-8=4y+4$;
subtracting (1) from this,

$$3x-13=8;\ \therefore x=\tfrac{21}{3}=7;\ \therefore (2),\ y=2x-3=11;$$

consequently the fraction is $\frac{7}{11}$.

[5.] Suppose $x$ bushels of rye and $y$ bushels of wheat; then, by the question,

$$x+y+28=100;\ \text{or},\ x+y=72\ \ldots\ (1).$$

Also the value of the barley in shillings is $65\frac{1}{3}$, that of the rye $3x$, and that of the wheat $4y$; so that the value of the whole 100 bushels is $3x+4y+65\frac{1}{3}$; that is, this expresses the number of shillings. But the number of shillings, at $3\frac{1}{3}$ per bushel, is $333\frac{1}{3}$; consequently,

$$3x+4y+65\frac{1}{3}=333\frac{1}{3};\ \text{or},\ 3x+4y=268\ \ldots\ (2);$$

subtracting 3 times (1) from this,

$$y=\ 52:\ \therefore (1);\ x=20;$$

therefore there must be 20 bushels of rye and 52 of wheat.

[6.] Let $x$ be the number of persons, and $y$ the number of shillings distributed; then the number of shillings each received was $\frac{y}{x}$; and by the question,

$$\frac{y}{x+3}=\frac{y}{x}-1;\ \therefore xy=xy+3y-x^2-3x$$

$$\frac{y}{x-2}=\frac{y}{x}+1;\ \therefore xy=xy-2y+x^2-2x;$$

transposing, $x^2+3x-3y=0\ \ldots\ (1)$

$$x^2-2x-2y=0\ \ldots\ (2);$$

$$\therefore 5x-y=0;\ \therefore y=5x;$$

substituting this in (2),

$$x^2-2x-10x=0;\ \therefore x-2-10=0;\ \therefore x=12;\ \therefore y=60;$$

hence the number of persons was 12, the sum distributed 60*s.* or £3, and consequently the share of each was 5*s.*

*Otherwise*, let $x$ be the number of persons, and $y$ the number of shillings each received; then the number of shillings distributed must have been $xy$. By the question, if the number of persons had been $x+3$, the share of each would have been $y-1$; and had the number been $x-2$, the share of each would have been $y+1$; consequently, since the number of persons, multiplied by the number of shillings each received, must in either case be the number of shillings distributed, we have the two equations,

$$(x+3)(y-1)=xy;\ \therefore xy+3y-x-3=xy,$$
$$(x-2)(y+1)=xy;\ \therefore xy+2y+x-2=xy;$$

transposing, $$3y-x=\ \ 3;\ \therefore x=3(y-1)$$
$$2y-x=-2;\ \therefore x=2(y+1);$$
$$\therefore 3y-3=2y+2;\ \therefore y=5;$$
$$\therefore x=2(y+1)=12;$$

hence the number of persons was 12, and the share of each was 5*s.*

[7.] Suppose *A* had $x$ shillings, and *B* $y$ shillings; then, by the question, *A* after the first sitting had $2x+4$, and consequently *B* had $y-x-4$; and *B* after the second sitting had $y-x-4+\frac{y}{2}$, so that in this sitting *A* lost $\frac{\ }{2}$ shillings. Also by the question,

$$2x+4=2(y-x-4)$$
$$y-x-4+\frac{y}{2}=3\left(2x+4-\frac{y}{2}\right);$$

transposing,

$$4x-2y=-12;\ \text{or, } 2x-y=-6;\ \therefore y=2x+6;$$
$$7x-3y=-16;$$

substituting the expression for $y$ in this,

$$7x-6x-18=-16;\ \therefore x=2;\ \therefore y=10;$$

consequently *A* began with 2*s.* and *B* with 10*s.*

[8.] Let $x$ represent the number of half-guineas, and $y$ the number of crowns; then, by the question, $3x=2y$. Also, $10\frac{1}{2}x+5y=144$, the number of shillings in the whole amount; hence we have the two equations,

$$3x-2y=0;\ \therefore x=\frac{2y}{3}$$
$$21x+10y=288;$$

substituting the expression for $x$ in the second equation,

$$14y+10y=288;\ \therefore 24y=288;\ \therefore y=12;$$
$$\therefore x=\frac{2y}{3}=8;$$

consequently there must be 8 half-guineas and 12 crowns.

[9.] Suppose *A*, *B*, *C* can do respectively, $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ of the whole in 1 day; then, by the question,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{15}$$

$$\text{sum, } 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{1}{12}+\frac{1}{20}+\frac{1}{15}=\frac{12}{60}=\frac{1}{5};$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10};$$

consequently all three working together can do $\frac{1}{10}$ of the whole in 1 day; therefore they can do the whole in 10 days.

[10.] Suppose the pipes *A*, *B*, *C*, fill respectively, $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ of the cistern in 1 minute; then, by the question,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{70}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{84}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{140}$$

$$\text{sum, } 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{1}{70}+\frac{1}{84}+\frac{1}{140}=\frac{14}{420}=\frac{1}{30};$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{60}\ \ldots (A).$$

From this equation subtract each of the first set, and there will result the three equations following, namely,

$$\frac{1}{x}=\frac{1}{60}-\frac{1}{140}=\frac{1}{105};\quad \frac{1}{y}=\frac{1}{60}-\frac{1}{84}=\frac{1}{210}$$

$$\frac{1}{z}=\frac{1}{60}-\frac{1}{70}=\frac{1}{420}.$$

These fractions are the parts of the whole cistern supplied by the respective pipes in 1 minute; consequently the pipe *A* fills it in 105 minutes, the pipe *B* in 210 minutes, and the pipe *C* in 420 minutes. Moreover, from (*A*), it appears that when all flow together, the fractional part filled in 1 minute is $\frac{1}{60}$; consequently, the whole will be filled in 60 min. or 1 hour.

It is observed in the book, that both this question and the preceding can be easily solved by help of only one unknown quantity: we shall here show how this is to be done.

Ex. 9. Suppose they do the work in $x$ days, then in 1 day they do $\frac{1}{x}$ part of it; but by the question,

$$\frac{1}{x}-\frac{1}{12}= \text{the part done by } C \text{ in 1 day;}$$
$$\frac{1}{x}-\frac{1}{20}= \ldots\ldots\ldots\ldots\ A \ldots\ldots\ldots$$
$$\frac{1}{x}-\frac{1}{15}= \ldots\ldots\ldots\ldots\ B \ldots\ldots\ldots$$
$$\therefore \frac{3}{x}-\frac{1}{5}=\frac{1}{-}, \text{ the part done by all in 1 day;}$$
$$\therefore \frac{2}{x}=\frac{1}{5}; \ \therefore x=10, \text{ the number of days.}$$

Ex. 10. Suppose the cistern is filled by all the pipes flowing together in $x$ minutes, then $\frac{1}{x}$ of it is filled in 1 minute; but by the question,

$$\frac{1}{x}-\frac{1}{70}= \text{the part filled by } C \text{ in 1 minute;}$$
$$\frac{1}{x}-\frac{1}{84} \ldots\ldots\ldots\ldots\ldots\ B \ldots\ldots\ldots\ldots$$
$$\frac{1}{x}-\frac{1}{140} \ldots\ldots\ldots\ldots\ldots\ A \ldots\ldots\ldots\ldots$$
$$\therefore \frac{3}{x}-\frac{1}{30}=\frac{1}{x}, \text{ the part filled by all in 1 minute;}$$
$$\therefore \frac{2}{x}=\frac{1}{30}; \ \therefore x=60, \text{ the number of minutes;}$$

consequently,

$$\frac{1}{60}-\frac{1}{70}=\frac{1}{420}, \text{ the part filled by } C \text{ in 1 minute;}$$

$$\frac{1}{60}-\frac{1}{84}=\frac{1}{210} \ldots\ldots\ldots\ldots\ldots B \ldots\ldots\ldots\ldots$$

$$\frac{1}{60}-\frac{1}{140}=\frac{1}{105} \ldots\ldots\ldots\ldots\ldots A \ldots\ldots\ldots\ldots$$

therefore the times of filling by $A$, $B$, $C$, separately, are 105 min., 210 min., 420 min.

The learner will understand that these two examples are solved by the aid of only one unknown quantity, not because the examples in the present section are, in general, such as *necessarily* to require the aid of more unknown quantities, but because the solutions, with a single unknown, are quite as easy as the solutions before given with three unknowns. In most of the examples here considered, there is no absolute necessity for more than one unknown quantity: the reason that more are employed is, that the work of an example is often much more difficult and perplexing, when the operator is limited to one unknown quantity, than when no restriction of this kind is imposed.

[11.] Let $x$ be the number of hours with the stream, and $y$ the number of hours against it; then, by the question,

$$x+y=10, \text{ and } 3x=2y.$$

From the first of these $x=10-y$; and substituting this in the second, we have,

$$30-3y=2y; \therefore 5y=30; \therefore y=6;$$
$$\therefore x=10-y=4;$$

consequently he was 4 hours rowing with the stream, and 6 hours rowing against it.

[12.] Suppose there were $x$ pounds of tin, and $y$ pounds of lead. By the question, the loss on 1 pound of tin, when weighed in water, is $\frac{5}{37}$ lb., and the loss on 1 pound of lead is $\frac{2}{23}$ lb.; so that the loss on $x$ pounds of tin and $y$ pounds of lead is $\frac{5x}{37}+\frac{2y}{23}$ pounds; but, by the question, this loss is 14 ℔.; hence,

$$x+y=120\,.\,.\,.\,.\,.\,(1)$$

$$\frac{5x}{37}+\frac{2y}{23}=14\,;$$

clearing fractions,

$$\begin{array}{rl} & 115x+\ \ 74y=11914\,.\,.\,.\,(2) \\ (1)\times 115, & \underline{115x+115y=13800} \\ & \text{diff.}\qquad 41y=\ \ 1886\,;\ \therefore y=\tfrac{1886}{41}=46\,; \\ & \therefore (1),\ x=120-y=74\,; \end{array}$$

consequently the composition contained 74 ℔. of tin and 46 ℔. of lead.

From this solution it appears that when a composition is formed of two known metals, the exact quantity of each metal in the compound can always be discovered. It was in some such way as this that Archimedes found the amount of alloy in the crown fabricated for Hiero, king of Syracuse. The artist was commissioned to make it of solid gold; but the king suspecting there was an alloy of silver in it, directed Archimedes to discover the cheat. The learner will easily see from this example how the extent of the fraud could be detected.

[13.] Let the number of pounds possessed by each be $x$, $y$, and $z$ respectively; then, by the question,

$$x+\frac{y}{2}+\frac{z}{2}=a$$

$$y+\frac{x}{3}+\frac{z}{3}=b$$

$$z+\frac{x}{4}+\frac{y}{4}=c$$

or, clearing fractions,

$$\begin{aligned} 2x+y+z&=2a\,.\,.\,.\,(1) \\ 3y+x+z&=3b\,.\,.\,.\,(2) \\ 4z+x+y&=4c\,.\,.\,.\,(3)\,; \end{aligned}$$

subtracting the first of these from the sum of the other two, we have,

$$3y+4z=3b+4c-2a\,;$$

and subtracting the second from the third,

$$3z-2y=4c-3b \quad \ldots (A);$$

adding this to the preceding,

$$y+7z=\ 8c-2a \ldots (B);$$

adding twice $(B)$ to $(A)$, $\quad 17z=20c-3b-4a$;

$$\therefore z=\tfrac{1}{17}(20c-3b-4a);$$

$$\therefore (B),\ y=8c-2a-7z=\tfrac{1}{17}(21b-4c-6a);$$

$$\text{and } (3),\ x=4(c-z)-y=\tfrac{1}{17}(22a-9b-8c);$$

which are the number of pounds possessed by $C$, $B$, and $A$ respectively.

[14.] The conditions of the question are,

$$\begin{aligned} 16a+4b+c&=42 \ldots (1)\\ 9a+3b+c&=22 \ldots (2)\\ 4a+2b+c&=\ 8 \ldots (3); \end{aligned}$$

subtracting (2) from (1), and (3) from (2), we have

$$\begin{aligned} 7a+b&=20 \ldots (A)\\ 5a+b&=14 \ldots (B)\\ \text{diff. } \overline{2a}\quad &= \overline{6};\ \therefore a=3; \end{aligned}$$

$\therefore (B)$, $b=14-5a=-1$; and (3), $c=2(4-2a-b)=-2$; these, therefore, are the values of the coefficients $a$, $b$, $c$.

[15.] This question appears difficult, and is very likely to puzzle a learner who sets about the solution of it without due consideration; but after a little reflection, it will be found to require but very little algebraical work. Let

$$v,\quad w,\quad x,\quad y,\quad z,$$

represent the number of pounds with which the persons commenced playing; these several sums being ranged according to their amounts, $v$ being the greatest of the above five numbers. It is plain that he who had the most money must have lost first, he whose money was next in amount lost second, and so on. As the first had £32 at the close of the fifth game, he must have had £16 at the close of the fourth, £8 at the close of the third, £4 at the close of the second, and £2 at the close of the first: we thus easily find what the first had immediately after his loss.

In like manner, since the second had £32 at the close of the fifth game, he must have had £16 at the close of the fourth game, £8 at the close of the third game, and £4 at the close of the second, that is, immediately after he lost. In the same way we find that the third must have had £8, the fourth £16, and the fifth £32, immediately after their respective losses. These results furnish us with the following conditions: namely,

$$\begin{array}{l} \left.\begin{array}{r} v-w-x-y-z=2 \\ 2w-2-2x-2y-2z=4 \end{array}\right\} \therefore \text{ subtracting twice the first from the second, } 4w-2v=2; \\ \begin{array}{r} 4x-4-4-4y-4z=8 \end{array} \quad \therefore v=2w-1 \\ 8y-8-8-8-8z=16 \\ 16z-16-16-16-16=32; \end{array}$$

dividing the last equation by 16, and transposing,

$$\begin{aligned} z &= 2+4=6; \\ \text{similarly, } y &= 2+3+6=11, \\ x &= 2+2+6+11=21, \\ w &= 2+1+6+11+21=41; \\ \text{and since } v &= 2w-1; \quad \therefore v=82-1=81; \end{aligned}$$

consequently, the respective sums at the beginning were £81, £41, £21, £11, and £6.

The learner will readily see the consistency of the preceding five equations with the conditions of the question, if he only keep before his mind the fact that, as the games proceed, those who have lost have each of them known sums, which are severally doubled at each subsequent game; and that those who have not yet lost have the unknown sums, arising from doubling the unknown sum at the outset, and then doubling the result, and so on till a loss takes place, when the sum each has left becomes known. These known sums furnish the right-hand members of the several equations of condition.*

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* The example solved in the text was taken from the collection of examples by the well-known German mathematician, Meyer Hirsch, where it is proposed but not worked. The book was translated into English by Mr. Wright, under the title of "Self-Examinations in Algebra." The example is one of the very few of any interest that has

[16.] Suppose the number of ounces to be taken from each ingot to be $x$, $y$, and $z$, respectively; then, by the question, the amount of silver in the portion $x$ is $\frac{7x}{16}$ oz., the amount of silver in the portion $y$ is $\frac{12y}{16}$ oz., and the amount of silver in the portion $z$ is $\frac{4z}{16}$ oz. In a similar manner, the amount of copper and of pewter in each portion is expressed; and as each expression will be a fraction, with 16 for denominator, it is better to conceive each expression to be multiplied by 16, and consequently to take 16 times the specified quantities in the composition: we thus have the three equations following, namely,

$$\begin{aligned} 7x + 12y + 4z &= 128 \ldots (1) \\ 3x + 3y + 7z &= 60 \ldots (2) \\ 6x + y + 5z &= 68 \ldots (3); \end{aligned}$$

subtracting the first from 4 times the second,

$$5x + 24z = 112 \ldots (A);$$

subtracting the second from 3 times the third,

$$15x + 8z = 144 \ldots (B);$$

subtracting $(B)$ from 3 times $(A)$, and then $(A)$ from 3 times $(B)$,

$$64z = 192; \therefore z = 3$$

$$40x = 320; \therefore x = 8;$$

consequently (3), $y = 68 - 6x - 5z = 5$;

therefore there must be 8 oz. taken from the first ingot, 5 oz. from the second, and 3 oz. from the third.

not, I believe, found its way till now into the school-books; and the Author of this Key has nowhere seen it solved, although it is probable that whatever solutions may have been given, they differ but little from that offered above. He may take this opportunity of mentioning that he is entirely responsible for all the solutions in this book; for although most of the examples were taken from various sources, many of them, no doubt, accompanied with their solutions, yet he preferred the labour of working, for this Key, every one of them himself, to the perhaps equally troublesome task of seeking out from among a number of volumes, few of which indeed are now in his possession, the solutions by others of the examples appropriated. He mentions this circumstance merely to account for the probable fact, that solutions may exist elsewhere having claims to a preference that some of these may not possess.

[17.] Let the four parts be $w, x, y, z$; then, by the question,

$$w+2=x-2=2y=\frac{z}{2};\ \text{and}\ w+x+y+z=90$$

$$\begin{aligned} w+2&=\frac{z}{2}\\ x-2&=\frac{z}{2}\\ y&=\frac{z}{4}\\ z&=z\\ \hline \text{Sum, } w+x+y+z&=\frac{9z}{4}=90; \end{aligned}$$

$$\therefore z=40;\ \therefore y=10,\ x=22,\ w=18;$$

consequently the parts are 18, 22, 10, and 40.

[18.] Let the first digit be $x$, the second $y$, and the third $z$; the local value of $x$ is $100x$, that of $y$ $10y$, and that of $z$ simply $z$, so that the number is $100x+10y+z$. By the question we have the following conditions, namely,

$$x-y=y-z,$$

$$\frac{100x+10y+z}{x+y+z}=26,$$

$$100x+10y+z+198=100z+10y+x;$$

adding $2y+z$ to each side of the first equation, we have $x+y+z=3y$; therefore, multiplying the second by $3y$, and transposing,

$$100x-68y+z=0\ \ldots\ (1)$$

$$99x-99z=-198;\ \therefore z-x=2\ \ldots\ (2);$$

adding this to the first equation, we have,

$$z-y=y-z+2;\ \therefore 2(z-y)=2;\ \therefore z-y=1\ \ldots\ (3);$$

$$\text{from (2), } z=x+2;\ \therefore \text{(3), } y=z-1=x+1;$$

substituting these expressions for $z$ and $y$ in (1),

$$100x-68x-68+x+2=0;$$

$$\therefore 33x=66;\ \therefore x=2;$$

$$\therefore y=x+1=3,\ \text{and}\ z=x+2=4;$$

consequently the number is 234.

[19.] Let $x$ be the number of men, and $y$ the number of women; then, by the question, the men have $px$ pence, and the women $qy$ pence; also, by the question,

$$x + y = n; \therefore x = n - y \ldots (1)$$
$$px + qy = 12a \ldots\ldots\ldots\ldots (2);$$

substituting (1) in (2),

$$np - (p - q)y = 12a;$$
$$\therefore y = \frac{np - 12a}{p - q}; \therefore x = n - y = \frac{12a - nq}{p - q};$$

the former expression denoting the number of women, and the latter the number of men.

[20.] At the first drawing, $\frac{b}{a}$ of the whole cask of wine is taken away, so that the part of the cask left is $1 - \frac{b}{a}$; of this quantity, $\frac{b}{a}$ is taken away at the second drawing, so $1 - \frac{b}{a}$ of it is left, that is $\left(1 - \frac{b}{a}\right)^2$; in the third drawing, $\frac{b}{a}$ of this is taken away, so that $1 - \frac{b}{a}$ of it, that is $\left(1 - \frac{b}{a}\right)^3$, is left, and so on; consequently, after $n$ drawings, the part of the whole cask of wine left is $\left(1 - \frac{b}{a}\right)^n$, that is, it is $\left(\frac{a - b}{a}\right)^n$; in other words, there is this part of $a$ gallons of wine in the cask; but $\left(\frac{a - b}{a}\right)^n$ of $a$ is $\frac{(a - b)^n}{a^{n-1}}$, which is therefore the number of gallons of wine left. The learner will observe that every $\frac{b}{a}$ part of the mixture taken away contains the $\frac{b}{a}$ part of the wine in that mixture, as well as the $\frac{b}{a}$ part of the water, both being equally diffused.

For some further particulars respecting the general theory of simultaneous equations of the first degree, see Note (A), at the end of this Key.

## THEORY OF EXPONENTS.—PAGE 149.

[1.]
$$\begin{array}{l}
x^{\frac{2}{3}}+x^{\frac{1}{3}}+1\\
x^{\frac{1}{3}}-1\\
\hline
x+x^{\frac{2}{3}}+x^{\frac{1}{3}}\\
\quad -x^{\frac{2}{3}}-x^{\frac{1}{3}}-1\\
\hline
x \qquad\qquad\quad -1\\
\hline
\end{array}$$

[2.]
$$\begin{array}{l}
a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}\\
a^{\frac{1}{4}}-b^{\frac{1}{4}}\\
\hline
a+a^{\frac{3}{4}}b^{\frac{1}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{3}{4}}\\
\quad -a^{\frac{3}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{3}{4}}-b\\
\hline
a \qquad\qquad\qquad\qquad\qquad -b\\
\hline
\end{array}$$

[3.]
$$\begin{array}{l}
x^{\frac{9}{4}}-x^{\frac{3}{2}}y^{\frac{3}{4}}+x^{\frac{3}{4}}y^{\frac{3}{2}}-y^{\frac{9}{4}}\\
x^{\frac{3}{4}}+y^{\frac{3}{4}}\\
\hline
x^{3}-x^{\frac{9}{4}}y^{\frac{3}{4}}+x^{\frac{3}{2}}y^{\frac{3}{2}}-x^{\frac{3}{4}}y^{\frac{9}{4}}\\
\quad x^{\frac{9}{4}}y^{\frac{3}{4}}-x^{\frac{3}{2}}y^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{9}{4}}-y^{3}\\
\hline
x^{3} \qquad\qquad\qquad\qquad\qquad -y^{3}\\
\hline
\end{array}$$

[4.]
$$\begin{array}{l}
15x^{-6}-\;\;7x^{-5}+6x^{-4}\\
8x^{-2}-\;\;3x^{-1}\\
\hline
120x^{-8}-\;56x^{-7}+48x^{-6}\\
\qquad\quad -\;45x^{-7}+21x^{-6}-18x^{-5}\\
\hline
120x^{-8}-101x^{-7}+69x^{-6}-18x^{-5}\\
\hline
\end{array}$$

[5.]
$$\begin{array}{l}
a^{2}-b\\
a^{\frac{1}{3}}-b^{-\frac{2}{3}}\\
\hline
a^{\frac{7}{3}}-a^{\frac{1}{3}}b-a^{2}b^{-\frac{2}{3}}+b^{\frac{1}{3}}\\
\hline
\end{array}$$

[6.]
$$\begin{array}{l}
(x^{\frac{1}{3}}-a^{\frac{1}{3}})^{2}=x^{\frac{2}{3}}-2a^{\frac{1}{3}}x^{\frac{1}{3}}+\;\;a^{\frac{2}{3}}\\
\qquad\qquad\qquad\; x^{\frac{1}{3}}-a^{\frac{1}{3}}\\
\hline
\qquad\qquad\qquad\; x-2a^{\frac{1}{3}}x^{\frac{2}{3}}+\;\;a^{\frac{2}{3}}x^{\frac{1}{3}}\\
\qquad\qquad\qquad\qquad -\;\;a^{\frac{1}{3}}x^{\frac{2}{3}}+2a^{\frac{2}{3}}x^{\frac{1}{3}}-a\\
\hline
\qquad\qquad\qquad\; x-3a^{\frac{1}{3}}x^{\frac{2}{3}}+3a^{\frac{2}{3}}x^{\frac{1}{3}}-a=\\
\qquad\qquad\qquad\; x-a-3a^{\frac{1}{3}}x^{\frac{1}{3}}(x^{\frac{1}{3}}-a^{\frac{1}{3}})=\\
\qquad\qquad\qquad\; x-a-3\sqrt[3]{ax}.\,(\sqrt{x}-\sqrt[3]{a}).
\end{array}$$

[7.]
$$\begin{array}{l}
x^{\frac{1}{3}}+y^{\frac{1}{4}}\\
x^{-\frac{1}{3}}+y^{-\frac{1}{4}}\\
\hline
1+x^{-\frac{1}{3}}y^{\frac{1}{4}}+x^{\frac{1}{3}}y^{-\frac{1}{4}}+1=x^{-\frac{1}{3}}y^{\frac{1}{4}}+x^{\frac{1}{3}}y^{-\frac{1}{4}}+2.
\end{array}$$

[8.]
$$\begin{array}{l}
x^{\frac{1}{2}}+y^{\frac{1}{2}}\,\big)\,x^{\frac{3}{2}}+2xy^{\frac{1}{2}}+2x^{\frac{1}{2}}y+y^{\frac{3}{2}}\big(\,x+(xy)^{\frac{1}{2}}+y\\
\qquad\qquad\; x^{\frac{3}{2}}+\;\;xy^{\frac{1}{2}}\\
\hline
\qquad\qquad\qquad\quad xy^{\frac{1}{2}}+2x^{\frac{1}{2}}y\\
\qquad\qquad\qquad\quad xy^{\frac{1}{2}}+\;\;x^{\frac{1}{2}}y\\
\hline
\qquad\qquad\qquad\qquad\qquad x^{\frac{1}{2}}y+y^{\frac{3}{2}}\\
\qquad\qquad\qquad\qquad\qquad x^{\frac{1}{2}}y+y^{\frac{3}{2}}\\
\hline
\end{array}$$

[9.] $$\begin{array}{l} x^{-\frac{1}{3}} - y^{-\frac{1}{3}}\big)\, x^{-1} - y^{-1}\big(x^{-\frac{2}{3}} + (xy)^{-\frac{1}{3}} + y^{-\frac{2}{3}} \\ \underline{x^{-1} - x^{-\frac{2}{3}}y^{-\frac{1}{3}}} \\ x^{-\frac{2}{3}}y^{-\frac{1}{3}} - y^{-1} \\ \underline{x^{-\frac{2}{3}}y^{-\frac{1}{3}} - x^{-\frac{1}{3}}y^{-\frac{2}{3}}} \\ x^{-\frac{1}{3}}y^{-\frac{2}{3}} - y^{-1} \\ \underline{x^{-\frac{1}{3}}y^{-\frac{2}{3}} - y^{-1}} \end{array}$$

[10.] $$\begin{array}{l} x^{-\frac{1}{2}} - y^{\frac{1}{2}}\big)\, x^{-\frac{3}{2}} - y^{\frac{3}{2}}\big(x^{-1} + x^{-\frac{1}{2}}y^{\frac{1}{2}} + y \\ \underline{x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}}} \\ x^{-1}y^{\frac{1}{2}} - y^{\frac{3}{2}} \\ \underline{x^{-1}y^{\frac{1}{2}} - x^{-\frac{1}{2}}y} \\ x^{-\frac{1}{2}}y - y^{\frac{3}{2}} \\ \underline{x^{-\frac{1}{2}}y - y^{\frac{3}{2}}} \end{array}$$

[11.] Sometimes division of compound surd expressions may be simplified by multiplying dividend and divisor by a suitable factor. Such multiplication of both by the same thing cannot have any effect on the quotient; it is merely introducing a common factor into both terms of a fraction. We shall adopt this expedient in the present example, multiplying each of the proposed expressions by $x^{\frac{1}{2}}y^{\frac{1}{2}}$, and then dividing as follows, reversing the terms of the divisor.

$$\begin{array}{l} x^{\frac{1}{2}}y^{-\frac{1}{2}}+1+x^{-\frac{1}{2}}y^{\frac{1}{2}}\big)\, x^{\frac{3}{2}}y^{-\frac{1}{2}}+2x+3x^{\frac{1}{2}}y^{\frac{1}{2}}+2y+x^{-\frac{1}{2}}y^{\frac{3}{2}}\big(x+(xy)^{\frac{1}{2}}+y \\ \underline{x^{\frac{3}{2}}y^{-\frac{1}{2}}+\ x+\ x^{\frac{1}{2}}y^{\frac{1}{2}}} \\ x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+2y \\ \underline{x+\ x^{\frac{1}{2}}y^{\frac{1}{2}}+\ y} \\ x^{\frac{1}{2}}y^{\frac{1}{2}}+\ y+x^{-\frac{1}{2}}y^{\frac{3}{2}} \\ \underline{x^{\frac{1}{2}}y^{\frac{1}{2}}+\ y+x^{-\frac{1}{2}}y^{\frac{3}{2}}} \end{array}$$

*Otherwise*, multiply dividend and divisor by $x$, $y$; then, reversing the terms of the divisor as above, the operation is as follows, the quantities being free of negative exponents:

$$\begin{array}{l} x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y\big)\, x^2+2x^{\frac{3}{2}}y^{\frac{1}{2}}+3xy+2x^{\frac{1}{2}}y^{\frac{3}{2}}+y^2\big(x+(xy)^{\frac{1}{2}}+y \\ \underline{x^2+\ x^{\frac{3}{2}}y^{\frac{1}{2}}+\ xy} \\ x^{\frac{3}{2}}y^{\frac{1}{2}}+2xy+2x^{\frac{1}{2}}y^{\frac{3}{2}} \\ \underline{x^{\frac{3}{2}}y^{\frac{1}{2}}+\ xy+\ x^{\frac{1}{2}}y^{\frac{3}{2}}} \\ xy+\ x^{\frac{1}{2}}y^{\frac{3}{2}}+y^2 \\ \underline{xy+\ x^{\frac{1}{2}}y^{\frac{3}{2}}+y^2} \end{array}$$

[12.] In this example, it will be better to divide by the divisor *twice*, than to take the square of the divisor, and then to seek the quotient by a single division: it will also be convenient to arrange the terms of the divisor thus, $x^{\frac{1}{3}}-y^{\frac{1}{3}}-x^{-\frac{1}{3}}y^{\frac{2}{3}}$, where the powers of $x$ regularly descend, as in the dividend.

$$
\begin{array}{l}
x^{\frac{1}{3}}-y^{\frac{1}{3}}-x^{-\frac{1}{3}}y^{\frac{2}{3}})\,x-3x^{\frac{2}{3}}y^{\frac{1}{3}}+5y-3x^{-\frac{2}{3}}y^{\frac{5}{3}}-x^{-1}y^{2}\,(x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{2}{3}}+2x^{-\frac{1}{3}}y+x^{-\frac{2}{3}}y^{\frac{4}{3}}\\
\quad\underline{x-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{2}{3}}}\\
\quad -2x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{2}{3}}+5y\\
\quad \underline{-2x^{\frac{2}{3}}y^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}+2y}\\
\qquad -x^{\frac{1}{3}}y^{\frac{2}{3}}+3y-3x^{-\frac{2}{3}}y^{\frac{5}{3}}\\
\qquad \underline{-x^{\frac{1}{3}}y^{\frac{2}{3}}+y+x^{-\frac{1}{3}}y^{\frac{4}{3}}}\\
\qquad\qquad 2y-x^{-\frac{1}{3}}y^{\frac{4}{3}}-3x^{-\frac{2}{3}}y^{\frac{5}{3}}\\
\qquad\qquad \underline{2y-2x^{-\frac{1}{3}}y^{\frac{4}{3}}-2x^{-\frac{2}{3}}y^{\frac{5}{3}}}\\
\qquad\qquad\quad x^{-\frac{1}{3}}y^{\frac{4}{3}}-x^{-\frac{2}{3}}y^{\frac{5}{3}}-x^{-1}y^{2}\\
\qquad\qquad\quad \underline{x^{-\frac{1}{3}}y^{\frac{4}{3}}-x^{-\frac{2}{3}}y^{\frac{5}{3}}-x^{-1}y^{2}}
\end{array}
$$

$$
\begin{array}{l}
x^{\frac{1}{3}}-y^{\frac{1}{3}}-x^{-\frac{1}{3}}y^{\frac{2}{3}})\,x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{2}{3}}+2x^{-\frac{1}{3}}y+x^{-\frac{2}{3}}y^{\frac{4}{3}}\,(x^{\frac{1}{3}}-y^{\frac{1}{3}}-x^{-\frac{1}{3}}y^{\frac{2}{3}}\\
\quad\underline{x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{2}{3}}}\\
\quad -x^{\frac{1}{3}}y^{\frac{1}{3}}+2x^{-\frac{1}{3}}y+x^{-\frac{2}{3}}y^{\frac{4}{3}}\\
\quad \underline{-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}+x^{-\frac{1}{3}}y}\\
\qquad -y^{\frac{2}{3}}+x^{-\frac{1}{3}}y+x^{-\frac{2}{3}}y^{\frac{4}{3}}\\
\qquad \underline{-y^{\frac{2}{3}}+x^{-\frac{1}{3}}y+x^{-\frac{2}{3}}y^{\frac{4}{3}}}
\end{array}
$$

It thus appears that the dividend is the cube of

$$x^{\frac{1}{3}}-x^{-\frac{1}{3}}y^{\frac{2}{3}}-y^{\frac{1}{3}}.$$

The preceding examples have been given for the purpose of affording sufficient practice to the learner in the management of fractional exponents; but in long operations, like that above, the continual recurrence of these exponents gives a confused appearance to the work. This may be avoided by adopting a plan which we shall here exemplify.

In the foregoing example, put $w$ for $x^{\frac{1}{3}}$, and $z$ for $y^{\frac{1}{3}}$, then the given divisor becomes changed into $w-z-w^{-1}z^{2}$, and the dividend into $w^{3}-3w^{2}z+5z^{3}-3w^{-2}z^{5}-w^{-3}z^{2}$, and the work stands as follows:

$$\begin{array}{l}
w-z-w^{-1}z^2)\,w^3-3w^2z+5z^3-3w^{-2}z^5-w^{-3}z^6\,(w^2-2wz-z^2+2w^{-1}z^3+w^{-2}z^4 \\
\qquad\qquad\quad \underline{w^3-w^2z-wz^2} \\
\qquad\qquad\qquad -2w^2z+wz^2+5z^3 \\
\qquad\qquad\qquad \underline{-2w^2z+2wz^2+2z^3} \\
\qquad\qquad\qquad\qquad\quad -wz^2+3z^3-3w^{-2}z^5 \\
\qquad\qquad\qquad\qquad\quad \underline{-wz^2+z^3+w^{-1}z^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2z^3-w^{-1}z^4-3w^{-2}z^5 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{2z^3-2w^{-1}z^4-2w^{-2}z^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad w^{-1}z^4-w^{-2}z^5-w^{-3}z^6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{w^{-1}z^4-w^{-2}z^5-w^{-3}z^6}
\end{array}$$

$$\begin{array}{l}
w-z-w^{-1}z^2)\,w^2-2wz-z^2+2w^{-1}z^3+w^{-2}z^4\,(w-z-w^{-1}z^2 \\
\qquad\qquad\quad \underline{w^2-wz-z^2} \\
\qquad\qquad\qquad -wz+2w^{-1}z^3+w^{-2}z^4 \\
\qquad\qquad\qquad \underline{-wz+z^2+w^{-1}z^3} \\
\qquad\qquad\qquad\qquad\quad z^2+w^{-1}z^3+w^{-2}z^4 \\
\qquad\qquad\qquad\qquad\quad \underline{z^2+w^{-1}z^3+w^{-2}z^4}
\end{array}$$

Putting now, in this last quotient, $x^{\frac{1}{3}}$ for $w$, and $y^{\frac{1}{3}}$ for $z$, it becomes $x^{\frac{1}{3}}-y^{\frac{1}{3}}-x^{-\frac{1}{3}}y^{\frac{2}{3}}$, as before. We may further remark here, that even negative exponents might have been avoided in the work, by multiplying the original dividend and divisor after the change here recommended, the former by $w^3$ and the latter by $w$, taking care to divide the *second* quotient when found by $w$, because the dividend will have been taken $w$ times what it ought to be. By applying this method, the foregoing dividend will be $w^6-3w^5z+5w^3z^3-3wz^5-z^6$, and the divisor will be $w^2-wz-z^2$. It is here seen that the powers of $z$ in the first expression will ascend regularly by unity, when the terms $0z^2$ and $0z^4$ are supplied, and that the second is complete as it is. We may therefore work by detached coefficients only, as follows:

$$w^2-w-1)\,w^6-3w^5+0+5w^3+0-3w-1\,($$

or still more simply, thus:

$$
\begin{array}{rrrrrrrrl}
1-1-1) & 1 & -3 & +0 & +5 & +0 & -3 & -1 & (1-2-1+2+1 \\
 & 1 & -1 & -1 & & & & & \\
\cline{2-4}
 & & -2 & +1 & +5 & & & & \\
 & & -2 & +2 & +2 & & & & \\
\cline{3-5}
 & & & -1 & +3 & +0 & & & \\
 & & & -1 & +1 & +1 & & & \\
\cline{4-6}
 & & & & 2 & -1 & -3 & & \\
 & & & & 2 & -2 & -2 & & \\
\cline{5-7}
 & & & & & 1 & -1 & -1 & \\
 & & & & & 1 & -1 & -1 & \\
\cline{6-8}
\end{array}
$$

$$
\begin{array}{rrrrrrl}
1-1-1) & 1 & -2 & -1 & +2 & +1 & (1-1-1 \\
 & 1 & -1 & -1 & & & \\
\cline{2-4}
 & & -1 & +0 & +2 & & \\
 & & -1 & +1 & +1 & & \\
\cline{3-5}
 & & & -1 & +1 & +1 & \\
 & & & -1 & +1 & +1 & \\
\cline{4-6}
\end{array}
$$

We have now to introduce the *descending* powers of $w$ into the terms of this last quotient, which then becomes $w^2-w-1$. Into these we have further to introduce the *ascending* powers of $z$; the quotient then becomes $w^2-wz-z^2$. Lastly, we have to divide this by $w$, the extra factor introduced into the dividend at the outset; we thus get $w-z-w^{-1}z^2$ for the true quotient; and replacing $w$ by $x^{\frac{1}{3}}$ and $z$ by $y^{\frac{1}{3}}$, we get, as before, the answer in the book.

From the three different ways in which the work of the foregoing example has been presented to the learner, he will readily perceive the great advantage of exercising a little address and ingenuity in dealing with complicated algebraical expressions. He will look in vain for simplifications like that above in the books; and if he wish to be thoroughly master of his subject, he must cultivate the habit of thinking for himself, and not deliver himself implicitly up to the guidance of formal rules. We would recommend him now to return to the first of the examples in the present article, and to solve them all conformably to the method here exhibited, avoiding fractional and negative exponents throughout.

## REDUCTION OF SURDS.—PAGE 152.

[1] $5\sqrt[3]{72}=5\sqrt[3]{(8.9)}=10\sqrt[3]{9}$.

[2.] $\sqrt[3]{(ax^3+bx^6)}=\sqrt[3]{x^3(a+bx^3)}=x\sqrt[3]{(a+bx^3)}$.

[3.] $\sqrt[3]{7(a^5-a^4x)}=\sqrt[3]{7a^3(a^2-ax)}=a\sqrt[3]{7(a^2-ax)}$.

[4.] $\sqrt{\frac{5}{7}a^3}=a\sqrt{\frac{5}{7}a}=a\sqrt{\frac{35}{49}a}=\frac{1}{7}a\sqrt{35a}$.

[5.] $\sqrt[3]{\frac{13}{7}}=\sqrt{\frac{13\times 49}{7^3}}=\frac{1}{7}\sqrt[3]{637}$.

[6.] $\sqrt{45a^3x^7}=\sqrt{(9.5a^2ax^6x)}=3ax^3\sqrt{5ax}$.

[7.] $\sqrt{\frac{ax}{4(a-x)}}=\frac{1}{2}\sqrt{\frac{ax(a-x)}{(a-x)^2}}=\frac{1}{2(a-x)}\sqrt{ax(a-x)}$.

[8.] $\frac{1}{7}\sqrt{\frac{2}{9}}=\frac{1}{7}\sqrt[3]{\frac{6}{27}}=\frac{1}{21}\sqrt[3]{6}$.

[9.] $\sqrt[m]{a^{m+n}x^{2m}}=\sqrt[m]{a^m a^n x^{2m}}=ax^2\sqrt[m]{a^n}$.

[10.] $\sqrt[m]{\frac{a^{\frac{m+n}{2}}}{x^{\frac{m}{2}}}}=\frac{a^{\frac{1}{2}}a^{\frac{n}{m}}}{x^{\frac{1}{2}}}=\left(\frac{a}{x}\right)^{\frac{1}{2}}\sqrt[m]{a^n}$.

[11.] $(a+x)\sqrt{\frac{a-x}{a+x}}=(a+x)\sqrt{\frac{a^2-x^2}{(a+x)^2}}=$
$\frac{a+x}{a+x}\sqrt{(a^2-x^2)}=\sqrt{(a^2-x^2)}$.

[12.] $\sqrt{(3a^2x+6abx+3b^2x)}=\sqrt{(a^2+2ab+b^2)3x}=$
$(a+b)\sqrt{3x}$.

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## ADDITION AND SUBTRACTION OF SURDS.
## PAGE 154.

[1.] $\sqrt{28}=\sqrt{(4.7)}=2\sqrt{7}$, and $\sqrt{63}=\sqrt{(9.7)}=3\sqrt{7}$;
$\therefore 3\sqrt{28}+2\sqrt{63}=6\sqrt{7}+6\sqrt{7}=12\sqrt{7}$.

[2.] $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{1}{3}\sqrt{6}$, and $\sqrt{\frac{1}{6}}=\sqrt{\frac{6}{36}}=\frac{1}{6}\sqrt{6}$;
$\therefore \frac{3}{4}\sqrt{\frac{2}{3}}-\frac{2}{5}\sqrt{\frac{1}{6}}=\left(\frac{3}{4}\cdot\frac{1}{3}-\frac{2}{5}\cdot\frac{1}{6}\right)\sqrt{6}=\frac{15-4}{60}\sqrt{6}=\frac{11}{60}\sqrt{6}$.

[3.] $\sqrt{27} = \sqrt{(9.3)} = 3\sqrt{3}$, and $\sqrt{48} = \sqrt{(16.3)} = 4\sqrt{3}$, and $\sqrt{147} = \sqrt{(49.3)} = 7\sqrt{3}$, and $\sqrt{75} = \sqrt{(25.3)} = 5\sqrt{3}$; $\therefore \sqrt{27} + \sqrt{48} + \sqrt{147} - 2\sqrt{75} = (3+4+7-10)\sqrt{3} = 4\sqrt{3}$.

[4.] $\sqrt{9a} = 3\sqrt{a}$; $\therefore 2x\sqrt{a} + 3x\sqrt{9a} - 10x\sqrt{a} = (2x + 9x - 10x)\sqrt{a} = x\sqrt{a}$.

[5.] $\sqrt[3]{2a^5x^2} = a\sqrt[3]{2a^2x^2}$, and $\sqrt[3]{2a^2x^5} = x\sqrt[3]{2a^2x^2}$;
$\therefore 7x\sqrt[3]{2a^5x^2} + 5a\sqrt[3]{2a^2x^5} - 3ax\sqrt[3]{2a^2x^2} =$
$(7ax + 5ax - 3ax)\sqrt[3]{2a^2x^2} = 9ax\sqrt[3]{2a^2x^2}$.

[6.] $\sqrt{a^3b} = a\sqrt{ab}$, and $\sqrt{ab^3} = b\sqrt{ab}$;
$\therefore \sqrt{a^3b} + \sqrt{ab^3} - b\sqrt{ab} = (a+b-b)\sqrt{ab} = a\sqrt{ab}$.

[7.] $\sqrt[4]{48a^4b^2c^4} = 2ac\sqrt[4]{3}\sqrt{b}$, and $\sqrt{bc^2} = c\sqrt{b}$;
$\therefore \sqrt[4]{48a^4b^2c^4} - a\sqrt{bc^2} = (2\sqrt[4]{3} - 1)ac\sqrt{b}$.

[8.] $\sqrt[3]{72} = \sqrt[3]{(8.9)} = 2\sqrt[3]{9}$, and $\sqrt[3]{\frac{1}{3}} = \sqrt[3]{\frac{9}{27}} = \frac{1}{3}\sqrt[3]{9}$;

$$\therefore \sqrt[3]{72} - 3\sqrt[3]{\frac{1}{3}} = (2-1)\sqrt[3]{9} = \sqrt[3]{9}.$$

[9.] $\sqrt{\frac{5}{3}} = \sqrt{\frac{15}{9}} = \frac{1}{3}\sqrt{15}$, and $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{1}{5}\sqrt{15}$;

$$\therefore \frac{1}{2}\sqrt{\frac{5}{3}} + \sqrt{15} - \frac{1}{4}\sqrt{\frac{3}{5}} = \left(\frac{1}{6} + 1 - \frac{1}{20}\right)\sqrt{15} = \frac{67}{60}\sqrt{15}.$$

[10.] $\sqrt{12} = \sqrt{(4.3)} = 2\sqrt{3}$, and $\sqrt{27} = \sqrt{(9.3)} = 3\sqrt{3}$,

$$\text{and } \sqrt{\frac{3}{16}} = \frac{1}{4}\sqrt{3};$$

$$\therefore \frac{3}{4}\sqrt{3} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}} =$$

$$\left(\frac{3}{4} - 1 + 12 - \frac{1}{2}\right)\sqrt{3} = 11\tfrac{1}{4}\sqrt{3}.$$

[11.] $\left(\frac{32a^5}{3b^4}\right)^{\frac{1}{3}} = \frac{a}{b}\left(\frac{4a^2}{3b}\right)^{\frac{1}{3}}$, and $\left(\frac{4a^2x^3}{81bc^3}\right)^{\frac{1}{3}} = \frac{x}{3c}\left(\frac{4a^2}{3b}\right)^{\frac{1}{3}}$,

$$\text{and } \left(\frac{16a^5}{12b^4}\right)^{\frac{1}{3}} = \left(\frac{4a^5}{3b^4}\right)^{\frac{1}{3}} = \frac{a}{b}\left(\frac{4a^2}{3b}\right)^{\frac{1}{3}}.$$

$$\therefore \left(\frac{32a^5}{3b^4}\right)^{\frac{1}{3}} + \left(\frac{4a^2x^3}{81bc^3}\right)^{\frac{1}{3}} - \left(\frac{16a^5}{12b^4}\right)^{\frac{1}{3}} = \left(\frac{2a}{b} + \frac{x}{3c} - \frac{a}{b}\right)\left(\frac{4a^2}{3b}\right)^{\frac{1}{3}} = \left(\frac{a}{b} + \frac{x}{3c}\right)\left(\frac{4a^2}{3b}\right)^{\frac{1}{3}}.$$

[12.] $\sqrt{80x^3} = \sqrt{(16.5x^3)} = 4x\sqrt{5x}$, and $\sqrt{45x^3} = \sqrt{(9.5x^3)} = 3x\sqrt{5x}$;

$$\therefore (a-x)\sqrt{5x} + \sqrt{80x^3} - \sqrt{45x^3} = (a-x+4x-3x)\sqrt{5x} = a\sqrt{5}\,x.$$

[13.] $\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$, and $\sqrt{\frac{3}{16}} = \frac{1}{4}\sqrt{3}$,

and $\sqrt{27} = \sqrt{(9.3)} = 3\sqrt{3}$;

$$\therefore 8\sqrt{\frac{3}{4}} - 2\sqrt{\frac{3}{16}} + 3\sqrt{27} = (4 - \frac{1}{2} + 9)\sqrt{3} = 12\frac{1}{2}\sqrt{3}.$$

[14.] $\sqrt[6]{4a^2} = \sqrt[3]{2a}$; $\therefore 6\sqrt[6]{4a^2} + 2\sqrt[3]{2a} = 8\sqrt[3]{2a}$.

[15.] $\sqrt{8} = 2\sqrt{2}$, $\sqrt{18} = 3\sqrt{2}$, $\sqrt{72} = 6\sqrt{2}$,

$$\therefore 4\sqrt{8} - 3\sqrt{18} - 5\sqrt{2} + \sqrt{72} = (8 - 9 - 5 + 6)\sqrt{2} = 0.$$

Again, $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{1}{5}\sqrt{15}$, $\sqrt{\frac{5}{3}} = \sqrt{\frac{15}{9}} = \frac{1}{3}\sqrt{15}$;

$$\therefore -\sqrt{15} + 2\sqrt{\frac{3}{5}} - \sqrt{\frac{5}{3}} = \left(-1 + \frac{2}{5} - \frac{1}{3}\right)\sqrt{15} = -\frac{14}{15}\sqrt{15}.$$

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## MULTIPLICATION AND DIVISION OF SURDS.

### PAGE 155.

[1.] $\sqrt{2} \times \sqrt[3]{3} = 2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}} = (8 \times 9)^{\frac{1}{6}} = \sqrt[6]{72}$.

[2.] $\sqrt[3]{2} \times \sqrt[6]{\frac{1}{5}} = 2^{\frac{1}{3}} \times \left(\frac{1}{5}\right)^{\frac{1}{6}} = 2^{\frac{2}{6}} \times \left(\frac{1}{5}\right)^{\frac{1}{6}} = \sqrt[6]{\frac{4}{5}} = \sqrt[6]{\frac{4.5^5}{5^6}} = \frac{1}{5}\sqrt[6]{(4.5^5)}$.

[3.] $\sqrt[3]{\frac{1}{2}} \times \sqrt[4]{\frac{1}{3}} = \left(\frac{1}{2}\right)^{\frac{1}{3}} \times \left(\frac{1}{3}\right)^{\frac{1}{4}} = \left(\frac{1}{2}\right)^{\frac{4}{12}}\left(\frac{1}{3}\right)^{\frac{3}{12}} = \left(\frac{1}{16} \cdot \frac{1}{27}\right)^{\frac{1}{12}} = \sqrt[12]{\frac{1}{432}}$; and the reciprocal of this is $\sqrt[12]{432}$.

As Tables of powers and roots seldom serve for exponents so high as 6, the final result in Ex. 2 should be regarded as $\sqrt[6]{\frac{4}{5}} = \sqrt[6]{\cdot 8}$, as it is more easy actually to compute this than the equivalent expression for it given above.

[4.] $\sqrt[3]{2} \times \sqrt[6]{\frac{1}{3}} \times \sqrt{3} = 2^{\frac{2}{6}}\left(\frac{1}{3}\right)^{\frac{1}{6}} 3^{\frac{3}{6}} = \left(4 \cdot \frac{1}{3} \cdot 27\right)^{\frac{1}{6}} =$
$\sqrt[6]{36} = \sqrt[3]{6}.$

[5.] $\left(\frac{a}{b}\right)^{\frac{1}{4}}\left(\frac{b}{a}\right)^{\frac{2}{4}} = \left(\frac{a}{b} \cdot \frac{b^2}{a^2}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{b}{a}} = \sqrt[4]{\frac{a^3 b}{a^4}} = \frac{1}{a}\sqrt[4]{a^3 b}.$

[6.] $\frac{\sqrt[5]{64}}{2} = \sqrt[5]{\frac{64}{32}} = \sqrt[5]{2}.$

[7.] $\frac{5}{2} \cdot \frac{12^{\frac{1}{3}}}{3^{\frac{1}{2}}} = \frac{5}{2}\left(\frac{144}{27}\right)^{\frac{1}{6}} = \frac{5}{2}\sqrt[6]{\frac{144 \times 27}{3^6}} = \frac{5}{6}\sqrt[6]{(144 \times 27)}.$

*Otherwise,* $\frac{5}{2} \cdot \frac{\sqrt[3]{12}}{\sqrt{3}} = 5\frac{\sqrt[3]{12}}{\sqrt{12}} = 5\frac{12^{\frac{2}{6}}}{12^{\frac{3}{6}}} = 5\sqrt[6]{\frac{12^2}{12^3}} = 5\sqrt[6]{\frac{1}{12}}$: this is the same in value as the expression just deduced.

[8.] $\sqrt{\frac{1}{2}} \times \frac{4}{3}\sqrt{\frac{6}{5}} = \frac{4}{3}\sqrt{\frac{3}{5}} = \frac{4}{15}\sqrt{15};$

and $2 \times \frac{4}{3}\sqrt{\frac{6}{5}} = \frac{8}{3}\sqrt{\frac{6}{5}} = \frac{8}{15}\sqrt{30};$

$\therefore \left(\sqrt{\frac{1}{2}} - 2\right) \div \frac{3}{4}\sqrt{\frac{5}{6}} = \frac{4}{15}(\sqrt{15} - 2\sqrt{30}).$

[9.] $\left(\sqrt{6} + \sqrt{2} - \sqrt[6]{\frac{3}{2}}\right)\sqrt{6} = 6 + \sqrt{12} - \sqrt[6]{\left(6^3 \times \frac{3}{2}\right)}$
$= 6 + 2\sqrt{3} - \sqrt[6]{(6^2 \times 3^2)}$
$= 6 + 2\sqrt{3} - \sqrt[3]{18}.$

[10.] $\frac{1}{2}(18)^{\frac{1}{6}}(2)^{\frac{3}{6}} = \frac{1}{2}(18.8)^{\frac{1}{6}} = \frac{1}{2}(6.3.4.2)^{\frac{1}{6}} = \frac{1}{2}\sqrt[3]{12}$;
and $\frac{1}{4}(8)^{\frac{1}{4}}(2)^{\frac{2}{4}} = \frac{1}{4}(8.4)^{\frac{1}{4}} = \frac{1}{4}(2^4.2)^{\frac{1}{4}} = \frac{1}{2}\sqrt[4]{2}$;

$$\therefore (\tfrac{1}{2}\sqrt[6]{18} + \tfrac{1}{4}\sqrt[4]{8})\sqrt{2} = \tfrac{1}{2}(\sqrt[3]{12} + \sqrt[4]{2}).$$

[11.] $\dfrac{\sqrt{18}}{\sqrt{3}} = \sqrt{6}$, and $2\dfrac{45^{\frac{1}{4}}}{3^{\frac{1}{2}}} = 2\left(\dfrac{45}{9}\right)^{\frac{1}{4}} = 2\sqrt[4]{5}$;

$$\therefore (\sqrt{18} - 2\sqrt[4]{45}) \div \sqrt{3} = \sqrt{6} - 2\sqrt[4]{5}.$$

[12.] $2\sqrt[3]{\left(\frac{2}{3}\cdot\frac{5}{6}\cdot 16\right)} = 2\sqrt[3]{\frac{80}{9}} = \frac{2}{3}\sqrt[3]{240} =$

$$\frac{2}{3}\sqrt[3]{(8.30)} = \frac{4}{3}\sqrt[3]{30};$$

$$\text{and } \left(\frac{30^2}{32}\right)^{\frac{1}{6}} = \left(\frac{1800}{64}\right)^{\frac{1}{6}} = \frac{1}{2}\sqrt[6]{1800};$$

$$\text{and } \frac{4}{3} \text{ of this is } \frac{2}{3}\sqrt[6]{1800}.$$

[13.] $\left(\dfrac{a^3y^3}{x^3}\cdot\dfrac{b^2x^2}{y^2}\cdot\dfrac{xy^2}{a^3b^2}\right)^{\frac{1}{6}} = y^{\frac{1}{2}} = \sqrt{y}.$

The number under the radical sign, in the result of Ex. 12, is $8 \times 225 = 2^3 \times 15^2$; so that this result may be replaced by $\frac{2}{3}\sqrt{2}\sqrt[3]{15}$, which is more readily computed by tables than the form above. It is in like manner easily seen that the number under the radical sign, in the first form of the result of Ex. 7, is $12^2 \times 3^3$; so that that form is immediately convertible into $\frac{5}{6}\sqrt[3]{12}\sqrt{3}$. In examples belonging to the present article, the object always is to express the results in forms, each of which shall involve but a single radical sign; but, as in these two instances, the actual computation of such results will often be facilitated by converting them into forms, each of which shall involve two or more radicals of lower degree, whenever the change is possible.

## Division by Binomial Surds.—Page 157

[1.]

$$\frac{3}{\sqrt{5}+\sqrt{7}}=\frac{3(\sqrt{5}-\sqrt{7})}{5-7}=\frac{3(\sqrt{5}-\sqrt{7})}{-2}=\frac{3}{2}(\sqrt{7}-\sqrt{5}).$$

[2.] $$\frac{a}{\sqrt{x}-\sqrt{y}}=\frac{a(\sqrt{x}+\sqrt{y})}{x-y}.$$

[3.]

$$\frac{5}{\sqrt[3]{5}-\sqrt[3]{2}}=\frac{5(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})}{5-2}=\frac{5}{3}(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}).$$

[4.] $$\frac{a}{\sqrt[3]{x}+\sqrt[3]{y}}=\frac{a(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})}{x+y}.$$

[5.] $$\frac{3+2\sqrt{7}}{\sqrt{7}-2}=\frac{(3+2\sqrt{7})(\sqrt{7}+2)}{7-4}\quad\frac{20+7\sqrt{7}}{3}$$

[6.] $$\frac{\sqrt{3}+1}{5-\sqrt{3}}=\frac{(\sqrt{3}+1)(5+\sqrt{3})}{25-3}=\frac{8+6\sqrt{3}}{22}=\frac{4+3\sqrt{3}}{11}.$$

[7.] $$\frac{2\sqrt{3}-5}{5-2\sqrt{3}}=\frac{(2\sqrt{3}-5)(5+2\sqrt{3})}{25-12}=\frac{-13}{13}=-1.$$

[8.] $$\frac{a}{\sqrt[4]{x}-\sqrt[4]{y}}=\frac{a(\sqrt[4]{x}+\sqrt[4]{y})}{\sqrt{x}-\sqrt{y}}=\frac{a(\sqrt[4]{x}+\sqrt[4]{y})(\sqrt{x}+\sqrt{y})}{x-y}.$$

[9.] $$\frac{2\sqrt{5}-3\sqrt{2}}{2\sqrt{2}+3\sqrt{5}}=\frac{(2\sqrt{5}-3\sqrt{2})(2\sqrt{2}-3\sqrt{5})}{8-45}=$$

$$\frac{13\sqrt{10}-42}{37}.$$

[10.] $$\frac{\sqrt{\frac{3}{5}}}{3\sqrt{3}-2\sqrt{2}}=\frac{(3\sqrt{3}+2\sqrt{2})\sqrt{\frac{3}{5}}}{27-8}=$$

$$\frac{3\frac{3}{\sqrt{5}}+2\sqrt{\frac{6}{5}}}{19}\qquad\frac{\frac{9}{5}\sqrt{5}+\frac{2}{5}\sqrt{30}}{19}.$$

*Otherwise,* $\frac{\sqrt{\frac{3}{5}}}{3\sqrt{3}-2\sqrt{2}} = \frac{\sqrt{3}}{3\sqrt{3}\sqrt{5}-2\sqrt{10}} =$

$$\frac{(3\sqrt{3}\sqrt{5}+2\sqrt{10})\sqrt{3}}{135-40} = \frac{9\sqrt{5}+2\sqrt{30}}{95},$$

and numerator and denominator of this divided by 5, gives the expression before obtained.

[11.] $\frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1}$, $\frac{1}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{1}$,

$$\frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{1}:$$

the sum of these is $2(\sqrt{2}+\sqrt{3})+3$, and $2(\sqrt{2}+\sqrt{3})$ subtracted from this sum leaves 3.

[12.] Reducing to a common denominator,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1} - \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2-(\sqrt{3}-1)^2}{2} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}.$$

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## SQUARE ROOTS OF BINOMIAL SURDS.

### PAGE 161.

THE formula for the solution of the examples in this article is,

$$\sqrt{(a \pm \sqrt{b})} = \sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}},$$

in which $c$ is put for $\sqrt{(a^2-b)}$.

[1.]

$$\sqrt{(7+2\sqrt{10})} = \sqrt{(7+\sqrt{40})} =$$

$$\sqrt{\frac{7+\sqrt{(49-40)}}{2}} + \sqrt{\frac{7-\sqrt{(49-40)}}{2}} = \sqrt{\frac{10}{2}} + \sqrt{\frac{4}{2}} =$$

$$\sqrt{5}+\sqrt{2}.$$

[2.] $\sqrt{(16-\sqrt{87})}$

$$=\sqrt{\frac{16+\sqrt{(256-87)}}{2}}-\sqrt{\frac{16-\sqrt{(256-87)}}{2}}$$

$$=\sqrt{\frac{16+13}{2}}-\sqrt{\frac{16-13}{2}}=\sqrt{\frac{29}{2}}-\sqrt{\frac{3}{2}}$$

$$=\frac{1}{2}(\sqrt{58}-\sqrt{6}).$$

[3.]

$$\sqrt{(10-\sqrt{96})}=\sqrt{\frac{10+\sqrt{(100-96)}}{2}}-\sqrt{\frac{10-\sqrt{(100-96)}}{2}}$$

$$=\sqrt{\frac{12}{2}}-\sqrt{\frac{8}{2}}=\sqrt{6}-2.$$

[4.]

$$\sqrt{(\tfrac{9}{4}-\sqrt{5})}=\sqrt{\frac{\frac{9}{4}+\sqrt{(\frac{81}{16}-5)}}{2}}-\sqrt{\frac{\frac{9}{4}-\sqrt{(\frac{81}{16}-5)}}{2}}$$

$$=\sqrt{\frac{\frac{9}{4}+\frac{1}{4}}{2}}-\sqrt{\frac{\frac{9}{4}-\frac{1}{4}}{2}}=\sqrt{\tfrac{5}{4}}-\sqrt{1}=\tfrac{1}{2}\sqrt{5}-1.$$

[5.]

$$\sqrt{(4+\sqrt{15\tfrac{3}{4}})}=\sqrt{\frac{4+\sqrt{(16-15\frac{3}{4})}}{2}}+\sqrt{\frac{4-\sqrt{(16-15\frac{3}{4})}}{2}}$$

$$=\sqrt{\frac{4+\frac{1}{2}}{2}}+\sqrt{\frac{4-\frac{1}{2}}{2}}=\tfrac{3}{2}+\tfrac{1}{2}\sqrt{7}=\tfrac{1}{2}(3+\sqrt{7}).$$

[6.]

$$1\frac{3}{8}+3\sqrt{\frac{1}{8}}=\frac{11}{8}+\frac{3}{4}\sqrt{2}=\frac{1}{8}(11+6\sqrt{2});\text{ and }\sqrt{(11+6\sqrt{2})}$$

$$=\sqrt{\frac{11+\sqrt{(121-72)}}{2}}+\sqrt{\frac{11-\sqrt{(121-72)}}{2}}$$

$$=\sqrt{\frac{11+7}{2}}+\sqrt{\frac{11-7}{2}}=\sqrt{9}+\sqrt{2}=3+\sqrt{2};$$

and this multiplied by $\frac{1}{\sqrt{8}}=\frac{1}{2\sqrt{2}}$, gives

$$\frac{3}{2\sqrt{2}}+\frac{1}{2}=\frac{3}{4}\sqrt{2}+{}_{2}.$$

[7.]

$$\frac{2}{3}-10\sqrt{\frac{1}{405}}=\frac{2}{405}(135-5\sqrt{405})\text{; and }\sqrt{(135-5\sqrt{405})}$$

$$=\sqrt{\frac{135+\sqrt{(135^2-25.405)}}{2}}-\sqrt{\frac{135-\sqrt{(135^2-25.405)}}{2}}$$

$$=\sqrt{\frac{135+90}{2}}-\sqrt{\frac{135-90}{2}}=\sqrt{\frac{225}{2}}-\sqrt{\frac{45}{2}};$$

and this multiplied by $\sqrt{\frac{2}{405}}$, gives

$$\sqrt{\frac{225}{405}}-\sqrt{\frac{45}{405}}=\sqrt{\frac{5}{9}}-\sqrt{\frac{1}{9}}=\frac{1}{3}(\sqrt{5}-1).$$

*Otherwise*, $\frac{2}{3}-10\sqrt{\frac{1}{405}}=\frac{2}{3}-\frac{10}{3}\sqrt{\frac{1}{45}}=\frac{2}{3}\left(1-5\sqrt{\frac{1}{45}}\right)$

$$=\frac{2}{9}\left(3-5\sqrt{\frac{1}{5}}\right)=\frac{2}{9}(3-\sqrt{5})\text{; and}$$

$$\sqrt{(3-\sqrt{5})}=\sqrt{\frac{3+\sqrt{(9-5)}}{2}}-\sqrt{\frac{3-\sqrt{(9-5)}}{2}}$$

$$=\sqrt{\frac{3+2}{2}}-\sqrt{\frac{3-2}{2}}=\sqrt{\frac{5}{2}}-\sqrt{\frac{1}{2}};$$

and this multiplied by $\sqrt{\frac{2}{9}}$, that is, by $\frac{1}{3}\sqrt{2}$, gives

$\frac{1}{3}(\sqrt{5}-1)$, as before.

[8.] $\frac{8}{15}+\frac{2}{\sqrt{15}}=\frac{2}{15}(4+\sqrt{15})$; and

$$\sqrt{(4+\sqrt{15})}=\sqrt{\frac{4+\sqrt{(16-15)}}{2}}+\sqrt{\frac{4-\sqrt{(16-15)}}{2}}$$

$$=\sqrt{\frac{4+1}{2}}+\sqrt{\frac{4-1}{2}}=\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}};$$

and this multiplied by $\sqrt{\frac{2}{15}}$, gives

$$\sqrt{\frac{1}{3}}+\sqrt{\frac{1}{5}}=\frac{1}{3}\sqrt{3}+\frac{1}{5}\sqrt{5}.$$

[9.] $$\sqrt{\{m+\sqrt{(m^2-n^2)}\}} = \sqrt{\frac{m+n}{2}} + \sqrt{\frac{m-n}{2}}$$
$$= \frac{1}{2}\{\sqrt{2(m+n)} + \sqrt{2(m-n)}\}.$$

[10.]

$$x + 2\sqrt{xy} + y = (\sqrt{x})^2 + 2\sqrt{xy} + (\sqrt{y})^2 = (\sqrt{x}+\sqrt{y})^2;$$
$$\therefore \sqrt{(x+y+2\sqrt{xy})} = \sqrt{x} + \sqrt{y}.$$

[11.] $x + \frac{1}{x} - \sqrt{(x^2+2)} = \frac{1}{x}\{x^2 + 1 - x\sqrt{(x^2+2)}\}$; and

$$\sqrt{\{x^2+1-x\sqrt{(x^2+2)}\}}$$
$$= \sqrt{\frac{x^2+1+\sqrt{(x^4+2x^2+1-x^4-2x^2)}}{2}}$$
$$-\sqrt{\frac{x^2+1-\sqrt{(x^4+2x^2+1-x^4-2x^2)}}{2}} = \sqrt{\frac{x+2}{2}} - \frac{x}{\sqrt{2}};$$

this, multiplied by $\frac{1}{\sqrt{x}}$, gives

$$\sqrt{\frac{x+2}{2x}} - \sqrt{\frac{x}{2}} = \frac{1}{2x}\sqrt{(2x^2+4x)} - \frac{1}{2}\sqrt{2x}.$$

[12.] $$\sqrt{\{3mn - 2m\sqrt{(3mn-m^2)}\}} =$$
$$\sqrt{\frac{3mn + \sqrt{(9m^2n^2 - 12m^3n + 4m^4)}}{2}}$$
$$-\sqrt{\frac{3mn - \sqrt{(9m^2n^2 - 12m^3n + 4m^4)}}{2}}$$
$$= \sqrt{\frac{3mn + 3mn - 2m^2}{2}} - \sqrt{\frac{3mn - 3mn + 2m^2}{2}}$$
$$= \sqrt{(3mn - m^2)} - m.$$

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## IMAGINARY OR IMPOSSIBLE QUANTITIES.—PAGE 166.

[1.] $$\begin{array}{r} -3\sqrt{-5} \\ -2\sqrt{-6} \\ \hline -6\sqrt{30}. \end{array}$$

[2.] $$\begin{array}{r} 2-\sqrt{-3} \\ \sqrt{-7} \\ \hline 2\sqrt{-7}+\sqrt{21}. \end{array}$$

[3.]

$$\begin{array}{l} 2+3\sqrt{-2} \\ 3-2\sqrt{-1} \\ \hline 6+9\sqrt{-2}-4\sqrt{-1}+6\sqrt{2}. \end{array}$$

[4.]

$$\begin{array}{l} a+b\sqrt{-1} \\ c+d\sqrt{-1} \\ \hline ac+(bc+ad)\sqrt{-1}-bd. \end{array}$$

[5.] $\frac{3\sqrt{-7}}{4\sqrt{-3}}=\frac{3}{4}\sqrt{\frac{-7}{-3}}=\frac{3}{4}\sqrt{\frac{7}{3}}=\frac{1}{4}\sqrt{21}.$

[6.] $\frac{2\sqrt{-5}}{-3\sqrt{-3}}=-\frac{2}{3}\sqrt{\frac{5}{3}}=-\frac{2}{9}\sqrt{15}.$

[7.] $\frac{2+\sqrt{-2}}{3-\sqrt{-3}}=\frac{(2+\sqrt{-2})(3+\sqrt{-3})}{9+3}$

$=\frac{1}{12}(6+2\sqrt{-3}+3\sqrt{-2}-\sqrt{6}).$

[8.] $\frac{4+\sqrt{-2}}{2-\sqrt{-3}}=\frac{(4+\sqrt{-2})(2+\sqrt{-3})}{4+3}$

$=\frac{1}{7}(8+2\sqrt{-2}+4\sqrt{-3}-\sqrt{6}).$

[9.] $\frac{3+2\sqrt{-1}}{3-2\sqrt{-1}}=\frac{(3+2\sqrt{-1})(3+2\sqrt{-1})}{13}$

$=\frac{9+12\sqrt{-1}-4}{13}=\frac{1}{13}(5+12\sqrt{-1}).$

[10.] $\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}=\frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{c^2+d^2}$

$=\frac{ac+bd+(bc-ad)\sqrt{-1}}{c^2+d^2}.$

[11.] $(a+b\sqrt{-1})^2=a^2+2ab\sqrt{-1}-b^2.$

[12.] $\sqrt{(a+b\sqrt{-1})}$

$=\sqrt{\frac{a+\sqrt{(a^2+b^2)}}{2}}+\sqrt{\frac{a-\sqrt{(a^2+b^2)}}{2}}$, by 160, Algebra.

## QUADRATIC EQUATIONS

### INVOLVING ONLY ONE UNKNOWN QUANTITY.

### PAGE 176.

[1.] Transposing, $x^2-5x=-6$;
completing the square, $x^2-5x+\left(\frac{5}{2}\right)^2=\frac{25}{4}-6=\frac{1}{4}$;
extracting the root, $x-\frac{5}{2}=\pm\frac{1}{2}$; $\therefore x=\frac{5\pm1}{2}=2$ or $3$.

[2.] Completing the square, $x^2-2x+1=16$;
extracting the root, $x-1=\pm4$; $\therefore x=1\pm4=-3$ or $5$.

[3.] Transposing, $x^2-\frac{1}{2}x=\frac{1}{2}$;

completing the square, $x^2-\frac{1}{2}x+\frac{1}{16}=\frac{1}{2}+\frac{1}{16}=\frac{9}{16}$;

extracting the root, $x-\frac{1}{4}=\pm\frac{3}{4}$; $\therefore x=\frac{1}{4}\pm\frac{3}{4}=-\frac{1}{2}$ or $1$.

[4.] Dividing by 6, $x^2+\frac{5}{6}x+\frac{1}{6}=0$;

transposing, $x^2+\frac{5}{6}x=-\frac{1}{6}$;

completing the square, $x^2+\frac{5}{6}x+\left(\frac{5}{12}\right)^2=\frac{25}{144}-\frac{1}{6}=\frac{1}{144}$;

extracting the root, $x+\frac{5}{12}=\pm\frac{1}{12}$;

$$\therefore x=-\frac{5\pm1}{12}=-\frac{1}{3}\text{ or }-\frac{1}{2}.$$

[5.] Multiplying by $x$, $x^2-2+6x=14$;
transposing, $x^2+6x=16$;
completing the square, $x^2+6x+9=16+9=25$;
extracting the root, $x+3=\pm5$; $\therefore x=-3\pm5=2$ or $-8$.

[6.] Multiplying by $x$, $4x^2=36-x+46x$;
transposing, $4x^2-45x=36$;

dividing by 4, $x^2-\frac{45}{4}x=9$;

completing the square, $x^2-\frac{45}{4}x+\left(\frac{45}{8}\right)^2=9+\frac{2025}{64}=\frac{2601}{64}$;

extracting the root, $x-\frac{45}{8}=\sqrt{\frac{2601}{64}}=\pm\frac{51}{8}$;

$$\therefore x=\frac{45\pm 51}{8}=12 \text{ or } -\frac{3}{4}.$$

[7.] Multiplying by $x$, $3x^2-169+3x=29x$;
transposing, $3x^2-26x=169$;

dividing by 3, $x^2-\frac{26}{3}x=\frac{169}{3}$;

completing the square, $x^2-\frac{26}{3}x+\left(\frac{13}{3}\right)^2=\frac{169}{3}+\frac{169}{9}=\frac{676}{9}$;

extracting the root, $x-\frac{13}{3}=\sqrt{\frac{676}{9}}=\pm\frac{26}{3}$;

$$\therefore x=\frac{13\pm 26}{3}=13 \text{ or } -\frac{13}{3}.$$

A little work might have been saved in this example from observing, in the step that completes the square, that

$$\left(\frac{13}{3}\right)^2+\frac{169}{3}=\left(\pm\frac{26}{3}\right)^2.$$

The learner will notice that we have put the double sign within the vinculum in the last form, but not in the preceding one. Why should not the preceding form be written $\left(\pm\frac{13}{3}\right)^2$? And if the double sign be inadmissible here, what claim has it to be introduced into the form stated to be the equivalent of the preceding expression?

These are very proper questions to be put, and the learner has a right to expect satisfactory answers to them. We reply as follows:—

The square has been completed by adding $\left(-\frac{13}{3}\right)^2$ to each side of the equation: the *result* of the operation indicated by the exponent is to be added to the $\frac{169}{3}$ on the second side of the equation: as this result is *plus*, there is no necessity to encumber the $\frac{13}{3}$ within the vinculum with the minus sign that really belongs to it; though, if the square root of the result spoken of were to be taken, it would be wrong to write that root otherwise than $-\frac{13}{3}$, since we are pre-informed that the result arose from squaring the number with the sign *minus* before it. When, however, the result of $\left(-\frac{13}{3}\right)^2$ is added to another number, $\frac{169}{3}$, we get a new number: and, in reference to *this*, nothing is stipulated or declared as to the *sign* of its square root; it has not been generated from any proposed root as the $\frac{169}{9}$ incorporated in it was; and, therefore, our business is simply to state from what root it *may* be generated, and it is clear that it may be generated either from the root $\frac{26}{3}$, or from the root $-\frac{26}{3}$. There is nothing *here*, as in the former case, to exclude one sign any more than the other; and this is why the ambiguous sign in $\left(\pm\frac{26}{3}\right)^2$ is introduced. In stating that $\left(\frac{13}{3}\right)^2+\frac{169}{3}=\left(\pm\frac{26}{3}\right)^2$, we merely affirm *equality as to result;* nothing is ever implied in the sign = as to the *generation* of the quantities between which it is placed; its office is limited to declaring an equality of resulting values. (See page 66, Algebra.)

[8.] Clearing fractions, using the L. C. M. 4 × 9.

$$9x^2-12x=4x^2-288x+2889;$$

transposing, $5x^2+276x=2889;$

dividing by 5, $x^2+\frac{276}{5}x=\frac{2889}{5}$;

completing the square,

$$x^2+\frac{276}{5}x+\left(\frac{138}{5}\right)^2=\frac{2889}{5}+\frac{19044}{25}=\frac{33489}{25};$$

extracting the root, $x+\frac{138}{5}=\sqrt{\frac{33489}{25}}=\pm\frac{183}{5}$;

$$\therefore x=\frac{-138\pm183}{5}=9, \text{ or } -64\frac{1}{5}.$$

[9.] Clearing fractions by multiplying by 20, the L. C. M. of the denominators,

$$10x^2+10x=4x^2-2x+13;$$

transposing, $6x^2+12x=13$;

dividing by 6, $x^2+2x=\frac{13}{6}$;

completing the square,

$$x^2+2x+1=\frac{13}{6}+1=\frac{19}{6};$$

extracting the root,

$$x+1=\sqrt{\frac{19}{6}}=\frac{1}{6}\sqrt{114};$$

$$\therefore x=-1\pm\frac{1}{6}\sqrt{114}.$$

[10.] Multiplying by $x+1$,

$$4x^2+4x-14+x=14x+14;$$

transposing, $4x^2-9x=28$;

dividing by 4, $x^2-\frac{9}{4}x=7$;

completing the square,

$$x^2-\frac{9}{4}x+\left(\frac{9}{8}\right)^2=7+\frac{81}{64}=\frac{529}{64};$$

extracting the root,

$$x-\frac{9}{8}=\sqrt{\frac{529}{64}}=\pm\frac{23}{8};$$

$$\therefore x=\frac{9\pm23}{8}=4 \text{ or } -1\frac{3}{4}.$$

[11.] Clearing fractions,

$$6x-8+x^2-6x+8=18x-72;$$

transposing, $x^2-18x=-72;$

completing the square,

$$x^2-18x+81=81-72=9;$$

extracting the root,

$$x-9=\pm3; \therefore x=9\pm3=6 \text{ or } 12.$$

[12.] Clearing fractions,

$$6x^2+16x-70+8x^2-20x-7x^2-35x=0;$$

transposing, $7x^2-39x=70;$

dividing by 7, $x^2-\frac{39}{7}x=10;$

completing the square,

$$x^2-\frac{39}{7}x+\left(\frac{39}{14}\right)^2=10+\frac{1521}{196}=\frac{3481}{196};$$

extracting the root, $x-\frac{39}{14}=\sqrt{\frac{3481}{196}}=\pm\frac{59}{14}$

$$\therefore x=\frac{39\pm59}{14}=7, \text{ or } -\frac{10}{7}.$$

[13.] Clearing fractions,

$$11x-x^2-24-4x+22=2x^2-10x+12;$$

transposing, $3x^2-17x=-14;$

dividing by 3, $x^2-\frac{17}{3}x=-\frac{14}{3};$

completing the square,

$$x^2-\frac{17}{3}x+\left(\frac{17}{6}\right)^2=\frac{289}{36}-\frac{14}{3}=\frac{121}{36};$$

extracting the root,

$$x-\frac{17}{6}=\sqrt{\frac{121}{36}}=\pm\frac{11}{6};$$

$$\therefore x=\frac{17\pm11}{6}=1, \text{ or } 4\frac{2}{3}.$$

[14.] Transposing, $\sqrt{2x} = 42 - 2x$;
squaring, $2x = 4(21 - x)^2 = 4(441 - 42x + x^2)$;

$$\therefore x = 882 - 84x + 2x^2;$$

transposing, $2x^2 - 85x = -882$;

dividing by 2, $x^2 - \frac{85}{2}x = -441$;

completing the square,

$$x^2 - \frac{85}{2}x + \left(\frac{85}{4}\right)^2 = \frac{7225}{16} - 441 = \frac{169}{16};$$

extracting the root,

$$x - \frac{85}{4} = \sqrt{\frac{169}{16}} = \pm\frac{13}{4};$$

$$\therefore x = \frac{85 \pm 13}{4} = 18, \text{ or } \frac{1}{2}(-7)^2.$$

The second of these values is, in result, the same as $\frac{49}{2}$: but if this be substituted for $x$ in the proposed irrational equation, it will not verify that equation, although it satisfies the rational equation deduced from it, and which, in fact, is the equation solved. The irrational term $\sqrt{2x}$ in the given equation has but a single value, namely $42 - 2x$; but the rational equation derived from it would have been the same if the proposed had been $\sqrt{2x} = \pm(42 - 2x)$: the given equation is, therefore, more restricted than that actually solved, the restriction being such that $\frac{49}{2}$ in the rational equation must be limited to the form $\frac{(-7)^2}{2}$ in the irrational equation. If each of the two roots, as above exhibited, be put for $x$ in the proposed equation, we have,

$$\sqrt{36} = 42 - 36; \text{ that is, } 6 = 6;$$

$$\text{and } \sqrt{(-7)^2} = 42 - (-7)^2; \text{ that is, } -7 = -7;$$

so that the equation is verified for both values of $x$: but if $\sqrt{49}$ had been taken with a *plus* sign, it is plain that the condition would not have been fulfilled. The learner must always bear in mind that the values of $x$ obtained for irrational equations, as derived not from solving *them*, but from solving rational equations deduced from them by involution, cannot be safely regarded as satisfying the original equations without some such modification as that above, and therefore that the values so derived should be always tested by the actual substitution of them in the proposed equations.

The equation which has given rise to these remarks may be otherwise and more easily and explicitly solved thus:—

Put $y$ for $\sqrt{2x}$, then the equation is $y^2+y=42$;

completing the square, $y^2+y+\frac{1}{4}=42+\frac{1}{4}=\frac{169}{4}$;

extracting the root, $$y+\frac{1}{2}=\sqrt{\frac{169}{4}}=\pm\frac{13}{2};$$

$$\therefore y=\frac{-1\pm 13}{2}=6, \text{ or } -7;$$

$$\therefore y^2=2x=36, \text{ or } (-7)^2;\ \therefore x=18, \text{ or } \frac{1}{2}(-7)^2.$$

Here the results are given at once in the admissible form, because the unambiguous character of the radical $\sqrt{2x}$ has not been obliterated by first squaring it, and then solving the deduced equation, as in the former method; and whenever we proceed in imitation of this plan, the final results will always preserve the necessary restrictions.

[15.] $$x^2+\sqrt{x}=\frac{72}{x}, \text{ or } x^3+\sqrt{x^3}=72.$$

Treating this equation after the manner of the second solution of the last, that is, putting $y$ for $\sqrt{x^3}$, and consequently $y^2$ for $x^3$, we have, $y^2+y=72$;

completing the square, $y^2+y+\frac{1}{4}=72+\frac{1}{4}=\frac{289}{4}$;

extracting the root, $$y+\frac{1}{2}=\sqrt{\frac{289}{4}}=\pm\frac{17}{2};$$

$$\therefore y=\frac{-1\pm 17}{2}=8, \text{ or } -9;$$

$$\therefore y^2=x^3=64, \text{ or } (-9)^2;\ \therefore x=4, \text{ or } \sqrt[3]{(-9)^2};$$

and both these forms satisfy the proposed irrational equation. The learner must thus see the advantage of solving irrational equations, without first rendering them rational in the usual way, whenever the means of doing so suggest themselves, as he is thereby saved the trouble of putting the results to the test by actual substitution.

[16.] Multiplying by 20, $12\sqrt{x}-40=x-5$;
transposing, $x-12\sqrt{x}=-35$;
putting $y$ for $\sqrt{x}$, and consequently $y^2$ for $x$,
$$y^2-12y=-35;$$
completing the square, $y^2-12y+36=1$;
extracting the root, $y-6=\pm 1$;
$$\therefore y=5 \text{ or } 7;\ \therefore y^2=x=5^2 \text{ or } 7^2.$$

These forms show that for the proposed equation to be satisfied, $\sqrt{x}$ must be either *plus* 5 or *plus* 7, not $\pm 5$ or $\pm 7$.

[17.] This equation is obviously the same as
$$(\sqrt{x})^3-2\sqrt{x}-(\sqrt{x})^2=0;$$
or, dividing by $\sqrt{x}$, it is, $(\sqrt{x})^2-\sqrt{x}=2$;
putting $y$ for $\sqrt{x}$, it is replaced by $y^2-y=2$;
completing the square, $y^2-y+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}$;
extracting the root, $y-\frac{1}{2}=\pm\frac{3}{2}$;
$$\therefore y=\frac{1\pm 3}{2}=2, \text{ or } -1;\ \therefore y^2=x=2^2, \text{ or } (-1)^2.$$

[18.] Dividing by $\sqrt{x}$ we have,
$$(\sqrt{x})^4+(\sqrt{x})^2=6;$$
that is, $x^2+x=6$;
completing the square,
$$x^2+x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4};$$
extracting the root, $x+\frac{1}{2}=\pm\frac{5}{2}$;
$$\therefore x=\frac{-1\pm 5}{2}=2, \text{or} -3.$$

[19.] Put $\sqrt{\frac{3x}{x+1}} = y$, then $\sqrt{\frac{x+1}{3x}} = \frac{1}{y}$;

and the equation is $y - \frac{1}{y} = 2$; $\therefore\ y^2 - 2y = 1$;

completing the square, $y^2 - 2y + 1 = 2$;
extracting the root, $y - 1 = \sqrt{2}$; $\therefore\ y = 1 \pm \sqrt{2}$;

$$\therefore\ y^2 = \frac{3x}{x+1} = (1 \pm \sqrt{2})^2 = 3 \pm 2\sqrt{2};$$

$$\therefore\ 3x = 3x + 3 \pm 2x\sqrt{2} \pm 2\sqrt{2};$$

$$\therefore\ \pm 2x\sqrt{2} = -3 \mp 2\sqrt{2};$$

$$\therefore\ x = \frac{-3 \mp 2\sqrt{2}}{\pm 2\sqrt{2}} = -1 \mp \frac{3}{2\sqrt{2}} =$$

$-1 \mp \frac{3}{4}\sqrt{2}$, or, which is the same, $-1 \pm \frac{3}{4}\sqrt{2}$.

[20.] Putting $y$ for $x^3$, the equation is replaced by
$y^2 - 7y = 8$;

completing the square, $y^2 - 7y + \frac{49}{4} = 8 + \frac{49}{4} = \frac{81}{4}$;

extracting the root, $y - \frac{7}{2} = \pm \frac{9}{2}$;

$\therefore\ y = \frac{7 \pm 9}{2} = 8$, or $-1$; $\therefore\ x^3 = 8$, or $-1$; $\therefore\ x = 2$, or $-1$.

[21.] Actually dividing numerator by denominator, the equation is $\sqrt{x} - 2 = x - 8$;
that is, putting $y$ for $\sqrt{x}$, and transposing,
$y^2 - y = 6$;

completing the square, $y^2 - y + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

extracting the root, $y - \frac{1}{2} = \pm \frac{5}{2}$;

$\therefore\ y = \frac{1 \pm 5}{2} = 3$, or $-2$; $\therefore\ y^2 = x = 3^2$, or $(-2)^2$.

[22.] Add 1 to both sides, the equation then becomes,

$$(2x^2+1)+\sqrt{(2x^2+1)}=12;$$

that is, putting $y$ for $\sqrt{(2x^2+1)}$, it is $y^2+y=12$;

completing the square, $y^2+y+\frac{1}{4}=12+\frac{1}{4}=\frac{49}{4}$;

extracting the root, $y+\frac{1}{2}=\pm\frac{7}{2}$;

$$\therefore y=\frac{1-\pm 7}{2}=3, \text{ or } -4; \therefore y^2=2x^2+1=3^2, \text{ or } (-4)^2.$$

We therefore have now to solve the two equations,

$$2x^2=8, \text{ and } 2x^2=(-4)^2-1;$$

$$\therefore x=\pm 2, \text{ and } x=\sqrt{\frac{(-4)^2-1}{2}}$$

This last value is, in its result, $\pm\frac{1}{2}\sqrt{30}$, which, however, must be considered as generated from the form here exhibited, in order that the proposed equation may be verified. Under this restriction as to the way in which the final result is produced, the expression for $2x^2+1$ is $(-4)^2$; and the first member of the proposed equation is therefore $15-4$, which is equal to the second member.

[23.] Multiplying by $\sqrt{\frac{x}{a}}$, the equation becomes

$$\frac{(x+a)x}{a}=(x-a); \therefore x^2+ax=ax-a^2;$$

$$\therefore x^2=-a^2; \therefore x=\pm a\sqrt{-1}.$$

[24.] Put $y$ for $\sqrt{x^n}$, then $y^2-2ay=b$;

completing the square, $y^2-2ay+a^2=a^2+b$;

extracting the root, $y-a=\sqrt{(a^2+b)}$;

$$\therefore y=a\pm\sqrt{(a^2+b)}; \therefore y^2=x^n=\{a\pm\sqrt{(a^2+b)}\}^2;$$

$$\therefore x=\{a\pm\sqrt{(a^2+b)}\}^{\frac{2}{n}}.$$

[25.] By the principle at page 113, Algebra,

$$\frac{\sqrt{(x^2-a^2)}}{x}=\frac{x-a}{x+a};$$

squaring, $$\frac{x^2-a^2}{x^2}=\left(\frac{x-a}{x+a}\right)^2;$$

dividing by $x-a$, and clearing fractions,

$$(x+a)^3=(x-a)x^2 \ldots (A);$$

that is, $x^3+3ax^2+3a^2x+a^3=x^3-ax^2$;
transposing, $4ax^2+3a^2x=-a^3$; $\therefore 4x^2+3ax=-a^2$;
dividing by 4, and completing the square,

$$x^2+\frac{3a}{4}x+\left(\frac{3a}{8}\right)^2=\frac{9a^2}{64}-a^2=-\frac{55a^2}{64};$$

extracting the root, $x+\frac{3a}{8}=\frac{a}{8}\sqrt{-55}$;

$$\therefore x=\frac{a}{8}(-3\pm\sqrt{-55}).$$

These are the values of $x$ deduced from the equation [$A$]; but this has arisen from dividing the equation which precedes it by $x-a$; so that if the factor thus removed be restored, the equation [$A$], after transposition, will then be

$$\{(x+a)^3-(x-a)x^2\}(x-a)=0.$$

This is of course satisfied, whichever of the two factors in the first member be equal to 0; the first of the two becomes 0 for the values of $x$ just obtained; the second becomes 0 for $x=a$: therefore the original equation is satisfied for this value of $x$, as well as for those values found above.

Whenever a factor, involving the unknown quantity, is removed from an equation, whatever values of the unknown are furnished by equating that factor to zero, belong also to the unknown in the original equation; and if a factor involving the unknown be introduced into an equation, and the equation be solved with this foreign factor incorporated, such of the roots as belong to the factor, when equated to zero, must be rejected as not belonging to the original equation.

[26.] This equation is the same as $x^2 - \frac{4}{9} = \frac{1}{x}\left(x + \frac{2}{3}\right)$; dividing each side by $x + \frac{2}{3}$, but remembering that as this is a factor containing $x$, the value of $x$ which renders it 0 must be a root of the equation, we have

$$x - \frac{2}{3} = \frac{1}{x}; \therefore 3x^2 - 2x = 3;$$

dividing by 3, $$x^2 - \frac{2}{3}x = 1;$$

completing the square, $x^2 - \frac{2}{3}x + \frac{1}{9} = 1 + \frac{1}{9} = \frac{10}{9}$;

extracting the root, $$x - \frac{1}{3} = \sqrt{\frac{10}{9}} = \frac{1}{3}\sqrt{10};$$

$$\therefore x = \frac{1}{3}(1 \pm \sqrt{10});$$

also, by equating to zero the factor suppressed at the outset,

$$x + \frac{2}{3} = 0; \therefore x = -\frac{2}{3};$$

therefore the values of $x$ which satisfy the proposed equation are $-\frac{2}{3}$, and $\frac{1}{3}(1 \pm \sqrt{10})$.

[27.] Put $y$ for $\sqrt[4]{(x+21)}$, then the equation is $y^2 + y = 12$; completing the square, $y^2 + y + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

extracting the root, $y + \frac{1}{2} = \pm\frac{7}{2}$;

$\therefore y = \frac{-1 \pm 7}{2} = 3$, or $-4$; $\therefore y^4 = (x+21) = 3^4$, or $(-4)^4$;

$\therefore x = 3^4 - 21 = 60$, or $(-4)^4 - 21$.

This last value is 235; but in order that it may satisfy the proposed equation, it must, when increased by 21, be regarded as produced from $(-4)^4$.

[28.] Adding 5 to each side, the equation becomes

$$x^2-2x+5+6\sqrt{(x^2-2x+5)}=16;$$

that is, putting $y^2$ for $x^2-2x+5$, $y^2+6y=16$;

completing the square, $y^2+6y+9=25$;

extracting the root, $y+3=\pm5$, $\therefore y=2$, or $-8$;

$$\therefore y^2=x^2-2x+5=4, \text{ or } (-8)^2;$$

taking the first of these values, we have

$$x^2-2x=-1\ldots(A);$$

completing the square, $x^2-2x+1=0$;

extracting the root, $x-1=0$; $\therefore x=1$;

taking the second of the above values of $y^2$, we have

$$x^2-2x=(-8)^2-5=59\ldots(B);$$

completing the square, $x^2-2x+1=60$;

extracting the root, $x-1=\sqrt{60}=\pm2\sqrt{15}$; $\therefore x=1\pm2\sqrt{15}$.

In reference to the last pair of values of $x$, it must be observed, in virtue of the condition $(B)$, that they cannot verify the proposed equation except under the restriction that $x^2-2x+5$ be regarded, not as 64, unconditionally, but as $(-8)^2$.

[29.] Adding 9 to each side, the equation is

$$2x^2+3x+9-5\sqrt{(2x^2+3x+9)}=6;$$

that is, putting $y^2$ for $2x^2+3x+9$, $y^2-5y=6$;

completing the square, $y^2-5y+\frac{25}{4}=6+\frac{25}{4}=\frac{49}{4}$;

extracting the root, $y-\frac{5}{2}=\pm\frac{7}{2}$; $\therefore y=\frac{5\pm7}{2}=6$, or $-1$;

$$\therefore y^2=2x^2+3x+9=36, \text{ or } (-1)^2;$$

taking the first of these values, we have

$$2x^2+3x=27\ldots(A);$$

dividing by 2, $x^2+\frac{3}{2}x=\frac{27}{2}$;

completing the square, $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{27}{2}+\frac{9}{16}=\frac{245}{16}$;

extracting the root, $x+\frac{3}{4}=\sqrt{\frac{245}{16}}=\pm\frac{7}{4}\sqrt{5}$;

$$\therefore x=\frac{1}{4}(-3\pm7\sqrt{5});$$

taking the other value of $y^2$, we have

$$2x^2+3x=(-1)^2-9=-8\ldots(B);$$

dividing by 2, $$x^2+\frac{3}{2}x=-4;$$

completing the square, $$x^2+\frac{3}{2}x+\frac{9}{16}=\frac{9}{16}-4=-\frac{55}{16};$$

extracting the root, $$x+\frac{3}{4}=\sqrt{-\frac{55}{16}}=\pm\frac{1}{4}\sqrt{-55};$$

$$\therefore x=\frac{1}{4}(-3\pm\sqrt{-55}):$$

And, in substituting this latter pair of values in the proposed, the condition [*B*] must be observed, otherwise the equation will not be satisfied; that is, $2x^2+3x+9$ must be regarded, not as 1 unconditionally, but as $(-1)^2$. If the proposed equation were rendered rational by involution, the roots would answer to the rational equation without any restriction. That equation would be one of the fourth degree, and this is the reason that there are *four* roots or values of $x$; it being a general principle that the number of roots of a *rational* equation is always equal to the number of its degree.

[30.] By transposing, $\sqrt{(4x+\sqrt{7x+2})}=1-2\sqrt{x};$

squaring, $4x+\sqrt{(7x+2)}=1-4\sqrt{x}+4x\ \ldots(1);$

$$\therefore \sqrt{(7x+2)}=1-4\sqrt{x};$$

squaring again, $7x+2=1-8\sqrt{x}+16x\ \ldots(2);$

$$\therefore 8\sqrt{x}=9x-1;$$

squaring again, $64x=81x^2-18x+1\ \ldots(A);$

transposing, $81x^2-82x=-1;$

dividing by 81, $x^2-\frac{82}{81}x=\frac{-1}{81};$

completing the square,

$$x^2-\frac{82}{81}x+\left(\frac{41}{81}\right)^2=\frac{1681}{81^2}-\frac{1}{81}=\frac{1600}{81^2};$$

extracting the root, $$x-\frac{41}{81}=\sqrt{\frac{1600}{81^2}}=\pm\frac{40}{81};$$

$$\therefore x=\frac{41\pm40}{81}=1 \text{ or } \frac{1}{81}.$$

As in this example, it is the rational equation (*A*), and not the original irrational equation that has been directly solved, we cannot be sure that either of these values applies to the proposed, till they are actually put to the test by substituting them each for $x$; for it is plain that the equation (1) would have been just the same though the original had been

$$\sqrt{(4x+\sqrt{7x+2})} = -1+2\sqrt{x};$$

and that (2) would have been just the same though the original had been

$$\sqrt{(4x-\sqrt{7x+2})} = \pm(1-2\sqrt{x});$$

and that (*A*), the equation actually solved, would have been just the same though the original had been of either of the forms, $\sqrt{(4x\mp\sqrt{7x+2})} = \pm(1\pm 2\sqrt{x})$, as is obvious from mere inspection.

Similar remarks apply to irrational equations in general, when the values supposed to satisfy their conditions are derived, as above, from the rational equations which result from clearing the radicals. These supposed values may all *fail* when put to the test of actual substitution in the given irrational equation, whenever the radicals in that equation are unambiguous as to sign. The roots brought out may belong, some to one, and some to another, of the different forms got, like those above, from modifying the signs of the irrational quantities which enter the original equation. These modified forms, in reference to the original one, are called CONGENERIC EQUATIONS, a name given to them by the late Mr. Horner, who first invited the attention of English algebraists to the failure of the supposed roots in certain cases; but the circumstance had been previously noticed by the French analysts, Garnier and Francœur, though nothing was done to furnish a satisfactory completion to this part of Algebra. The writer of these remarks hopes that the illustrations and explanations, given in the course of the foregoing solutions, will tend to remove all doubt and obscurity as to the correct theory of irrational equations, as far, at least, as the subject can with propriety be treated in a work of this kind.

## QUADRATIC EQUATIONS SOLVED BY RULE II.—PAGE 179.

There are a few examples given at page 179 of the Algebra, to be worked independently of the method of completing the square.

This neat and compact mode of solution we would strongly recommend to the learner, as that which should always be employed whenever it is foreseen that fractions would be introduced by the application of Rule I. In all such cases it will save time and figures, and will thus diminish the risk of error in the work. In the solution of the foregoing examples, the old rule has been uniformly applied, because long and universal practice has given a sanction and a currency to it that renders it indispensable, in an elementary writer on Algebra, to present it in the fullest detail to his readers; and, moreover, because a familiarity with the common method is of advantage to the clear understanding of this. In future, however, we shall always apply to the solution of a quadratic the simpler process of Rule II., except in those comparatively few cases in which fractions cannot occur in the former mode of operation; and even in such cases we shall sometimes use it, though no saving of figures may be effected.

The solutions of the examples at page 179, according to the method here recommended, are as follow.

[6.] By Rule II, the simple equation derived from this is,

$$6x-2=\surd(96+4)=\pm 10;$$

$$\therefore x=\frac{2\pm 10}{6}=2, \text{ or } -1\frac{1}{3}.$$

[7.] The derived simple equation is,

$$10x+7=\surd(1320+49)=\surd 1369=\pm 37;$$

$$\therefore x=\frac{-7\pm 37}{10}=3, \text{ or } -\frac{22}{5}=-4\frac{2}{5}.$$

[8.] The derived simple equation is,

$$2x-9=\surd(-56+81)=\pm 5;$$

$$\therefore x=\frac{9\pm 5}{2}=7, \text{ or } 2.$$

[9.] The simple equation derived from this is,

$$x-3=\sqrt{(7+9)}=\pm 4;$$

$$\therefore x=3\pm 4=7, \text{ or } -1.$$

[10.] The simple equation is,

$$12x+5=\sqrt{(24+25)}=\pm 7;$$

$$\therefore x=\frac{-5\pm 7}{12}=\frac{1}{6}, \text{ or } -1.$$

[11.] The simple equation derived from this is,

$$14x-3=\sqrt{(21+9)}=\sqrt{30};$$

$$\therefore x=\frac{3\pm\sqrt{30}}{14}.$$

[12.] The derived simple equation is,

$$20x+11=\sqrt{(120+121)}=\sqrt{241};$$

$$\therefore x=\frac{-11\pm\sqrt{241}}{20}.$$

[13.] The derived simple equation is,

$$10x-7=\sqrt{(-60+49)}=\sqrt{-11};$$

$$\therefore x=\frac{7\pm\sqrt{-11}}{10}.$$

The learner will perceive that the peculiarity of the foregoing method of solution which most strongly recommends it to general adoption is, that whenever fractions are absent from the proposed coefficients, they must be absent from the entire process, for the work never introduces them: and sometimes even when fractions are present in the given numbers, the first step of the operation converts them into integers; as, for instance, in examples 9 and 11 above.

## QUESTIONS TO BE SOLVED BY QUADRATIC EQUATIONS WITH ONLY ONE UNKNOWN QUANTITY.—PAGE 184.

[5.] Let $x$ represent the greater number, then by the question, $x-9$ represents the less: also, by the question,

$$(2x-9)x=266\text{; that is, } 2x^2-9x=266;$$

the simple equation derived from this by Rule II. is,

$$4x-9=\sqrt{(2128+81)}=\sqrt{2209}=\pm 47;$$

$$\therefore x=\frac{9\pm 47}{4}=14, \text{ or } -9\frac{1}{2};$$

$$\therefore x-9=5, \text{ or } -18\frac{1}{2};$$

consequently the two numbers are either 14 and 5, or $-9\frac{1}{2}$ and $-18\frac{1}{2}$.

[6.] Let $x$ be one of the numbers, then by the question, $41-x$ is the other: also, by the question,

$$x^2+(41-x)^2=901;$$

that is, $2x^2-82x+1681=901$, or $x^2-41x=-390$;

$\therefore$ Rule II., $2x-41=\sqrt{(-1560+41^2)}=\sqrt{121}=\pm 11$;

$$\therefore x=\frac{41\pm 11}{2}=15, \text{ or } 26;$$

$$\therefore 41-x=26, \text{ or } 15.$$

Hence there is but one pair of numbers that can fulfil the required conditions, namely, the numbers 15 and 26.

[7.] Let one of the parts be $x$, then by the question, the other part will be $20-x$: also

$$20(20-x)=x^2, \text{ or } 400-20x=x^2;$$

that is, by transposing, $x^2+20x=400$;

$\therefore$ Rule II., $2x+20=\sqrt{(1600+400)}=20\sqrt{5}$;

$$\therefore x=\frac{-20\pm 20\sqrt{5}}{2}=-10\pm 10\sqrt{5};$$

$$\therefore 20-x=30\mp 10\sqrt{5};$$

consequently, the parts are either $-10+10\sqrt{5}$ and $30-10\sqrt{5}$, or $-10-10\sqrt{5}$ and $30+10\sqrt{5}$.

[8.] Let $x$ be the number of sheep bought, then by the question, $x-15$ is the number sold: also, by the question, the purchase-price of each in shillings is $\frac{1200}{x}$, and the selling-price of the whole $x-15$ is,

$$(x-15)\left(\frac{1200}{x}+2\right)=1080 \ldots (A);$$

multiplying by $\frac{x}{2}$, $(x-15)(600+x)=540x$;

$$\text{that is,} \quad x^2+585x-9000=540x;$$
$$\text{transposing,} \quad x^2+45x=9000;$$
$$\therefore \text{Rule II., } 2x+45=\sqrt{(36000+45^2)}=\sqrt{38025}=\pm 195;$$
$$\therefore x=\frac{-45\pm 195}{2}=75, \text{ or } -120;$$

consequently the number bought was 75; the $-120$, although satisfying the algebraical condition $(A)$, being excluded from the nature of the question: also the cost-price of each was $\frac{1200}{75}$ s., that is, 16s.

[9.] Suppose there were $x$ yards in the finer piece, then by the question, there were $x+2$ yards in the coarser piece: also, by the question, the cost of one yard of the finer, in pounds, was $\frac{18}{x}$, and the cost of one yard of the coarser $\frac{16}{x+2}$: the former cost is stated to be 4s. or £$\frac{1}{5}$ more than this; therefore

$$\frac{18}{x}=\frac{16}{x+2}+\frac{1}{5};$$

$$\text{clearing fractions,} \quad 90x+180=80x+x^2+2x;$$
$$\text{transposing,} \quad x^2-8x=180 \ldots (A);$$
$$\therefore \text{Rule II., } 2x-8=\sqrt{(720+64)}=\sqrt{784}=\pm 28;$$
$$\therefore x=\frac{8\pm 28}{2}=18, \text{ or } -10;$$

consequently there were 18 yards in the finer piece, and $x+2$, that is 20 yards, in the coarser. The price per yard of the former was £$\frac{18}{x}$, that is £1; and the price per yard of the latter was £$\frac{16}{x+2}$, that is £$\frac{16}{20}$, or 16*s*.

The solution of equation (*A*) in this example, seeing that no fractions can be introduced by completing the square, would have been fully as convenient, if not more so, by Rule I.; and although, as a *general* method, Rule II. is decidedly preferable, yet when no fractions are introduced by the other rule, it may be a little shorter to apply it as noticed at p. 140. By completing the square, (*A*) becomes $x^2-8x+16=196$; $\therefore$ extracting the root $x-4=\pm 14$, and $x=18$, or $-10$ as above.

[10.] Let $x$ be the number, then by the question,

$$5\sqrt[4]{x}-3\sqrt{x}=\frac{4}{3};$$

that is, putting $y$ for $\sqrt[4]{x}$, and multiplying by 3,

$$15y-9y^2=4;\ \text{or}\ 9y^2-15y=-4;$$

$$\therefore \text{Rule II., } 18y-15=\sqrt{(-144+15^2)}=\pm 9;$$

$$\therefore y=\frac{15\pm 9}{18}=\frac{4}{3},\ \text{or}\ \frac{1}{3};$$

$$\therefore y^4=x=\frac{256}{81},\ \text{or}\ \frac{1}{81};\ \text{that is, } 3\frac{13}{81},\ \text{or}\ \frac{1}{81}.$$

[11.] Let $x$ be the number of men, then by the question, $20-x$ is the number of women: also, by the question, each man receives $\frac{24}{x}$ pounds, and each woman $\frac{24}{20-x}$; but the former number is by the question 1 more than the latter; therefore,

$$\frac{24}{x}=\frac{24}{20-x}+1\ldots(A);$$

clearing fractions, $480-24x=24x+20x-x^2$;
transposing, $x^2-68x=-480$;
completing the square,

$$x^2-68x+34^2=1156-480=676;$$

extracting the root, $x-34=\sqrt{676}=\pm 26$;

$$\therefore x=34\pm 26=8,\ \text{or}\ 60;$$

$$\therefore 20-x=12,\ \text{or}\ -40.$$

These results show that the algebraical condition $(A)$ is satisfied for either $x=8$ or $x=60$; but the restrictions of the *question* forbid the latter value, because, as we see, if 60 were admitted as the number of men, then the number of women, to make the number of both together 20, must be $-40$, which is absurd; consequently the number of men is 8 exclusively, and the number of women 12.

[12.] Let one of the parts be $x$, then by the question, the other part must be $39-x$: also, by the question,

$$x^3+(39-x)^3=17199;$$

that is, by actually cubing, as follows,

$$\begin{array}{rrrr}
 & 39- & x & \\
 & 39- & x & \\
\hline
 & 351- & 39x & \\
 & 117- & 39x+ & x^2 \\
\hline
 & 1521- & 78x+ & x^2 \\
 & 39- & x & \\
\hline
 & 13689- & 702x+ & 39x^2 \\
 & 4563- & 2340x & \\
 & & -1521x+ & 78x^2-x^3 \\
\hline
x^3+ & 59319- & 4563x+ & 117x^2-x^3=17199;
\end{array}$$

$$\begin{aligned}
&\text{transposing,} && 117x^2-4563x=-42120;\\
&\text{dividing by } 9, && 13x^2-\ \ 507x=-4680;\\
&\text{dividing by } 13, && x^2-\ \ \ 39x=-360;
\end{aligned}$$

$\therefore$ Rule II., $2x-39=\sqrt{(-1440+39^2)}=\sqrt{81}=\pm 9;$

$$\therefore x=\frac{39\pm 9}{2}=15, \text{ or } 24;$$

$$\text{and } 39-x=24, \text{ or } 15;$$

consequently the two parts are 15 and 24.

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## EQUATIONS OF THE THIRD AND FOURTH DEGREES.—PAGE 193.

[1.] Multiplying by $x$, $x^4-2x^3+3x^2-2x=0\ldots(A)$; the factors of the first member are,

$$\begin{aligned}
&x^2\pm\sqrt{(2x^3-3x^2+2x)}\\
&=x^2-x\pm\sqrt{(-2x^2+2x)}\\
&=x^2-x+1\pm\sqrt{(1)};
\end{aligned}$$

therefore the two factors of $(A)$ are $(x^2-x)(x^2-x+2)$; and consequently, dividing by $x$, the proposed equation is $(x-1)(x^2-x+2)=0$.

[2.] The factors of $x^4-2x^3+3x^2-2x+k$ are,

$$x^2\pm\sqrt{(2x^3-3x^2+2x-k)}$$
$$=x^2-x\pm\sqrt{(-2x^2+2x-k)}$$
$$=x^2-x+1\pm\sqrt{(1-k)};$$

consequently the proposed equation is,

$$(x^2-x+1+\sqrt{1-k})(x^2-x+1-\sqrt{1-k})=0.$$

[3.] The factors of $x^4+4x^3+4x^2-k$, are

$$x^2\pm\sqrt{(-4x^3-4x^2+k)}$$
$$=x^2+2x\pm\sqrt{(k)};$$

consequently the proposed equation is,

$$(x^2+2x+\sqrt{k})(x^2+2x-\sqrt{k}=0.$$

[4.] The factors of $x^4-4x^3+2x^2+4x+k$ are,

$$x^2\pm\sqrt{(4x^3-2x^2-4x-k)}$$
$$=x^2-2x\pm\sqrt{(2x^2-4x-k)}$$
$$=x^2-2x-1\pm\sqrt{(1-k)};$$

consequently the proposed equation is,

$$(x^2-2x-1+\sqrt{1-k})(x^2-2x-1-\sqrt{1-k})=0.$$

[5.] The factors of $x^4+6x^3+x^2-24x+k$ are,

$$x^2\pm\sqrt{(-6x^3-x^2+24x-k)}$$
$$=x^2+3x\pm\sqrt{(8x^2+24x-k)}$$
$$=x^2+3x-4\pm\sqrt{(16-k)};$$

consequently the proposed equation is,

$$(x^2+3x-4+\sqrt{16-k})(x^2+3x-4-\sqrt{16-k})=0.$$

[6.] The factors of $x^4-6x^3+5x^2+12x+k$ are,

$$x^2\pm\sqrt{(6x^3-5x^2-12x-k)}$$
$$=x^2-3x\pm\sqrt{(4x^2-12x-k)}$$
$$=x^2-3x-2\pm\sqrt{(4-k)};$$

consequently the proposed equation is,

$$(x^2-3x-2+\sqrt{4-k})(x^2-3x-2-\sqrt{4-k})=0.$$

[7.] Dividing the polynomial $8x^4-16x^3+24x^2-2x+\frac{1}{8}$ by 2, we have $4x^4-8x^3+12x^2-x+\frac{1}{16}$;
and the factors of this are,

$$\begin{aligned} &2x^2\pm\sqrt{\left(8x^3-12x^2+x-\frac{1}{16}\right)} \\ &=2x^2-2x\pm\sqrt{\left(-8x^2+x-\frac{1}{16}\right)} \\ &=2x^2-2x+\frac{1}{4}\pm\sqrt{(-7x^2)}; \end{aligned}$$

consequently the proposed equation is,

$$\left\{2x^2-(2-\sqrt{-7})x+\frac{1}{4}\right\}\left\{2x^2-(2+\sqrt{-7})x+\frac{1}{4}\right\}=0.$$

[8.] The factors of $x^4+8x^3-5x^2-24x+9$ are,

$$\begin{aligned} &x^2\pm\sqrt{(-8x^3+5x^2+24x-9)} \\ &=x^2+4x\pm\sqrt{(21x^2+24x-9)} \\ &=x^2+4x-3\pm\sqrt{(15x^2)}; \end{aligned}$$

consequently the proposed equation is,

$$\{x^2+(4-\sqrt{15})x-3\}\{x^2+(4+\sqrt{15})x-3\}=0.$$

[9.] The factors of $x^4-3x^3+7x^2+6x+4$ are,

$$\begin{aligned} &x^2\pm\sqrt{(3x^3-7x^2-6x-4)} \\ &=x^2-\frac{3}{2}x\pm\sqrt{\left(-\frac{19}{4}x^2-6x-4\right)} \\ &=x^2-\frac{3}{2}x-2\pm\sqrt{\left(-\frac{35}{4}x^2\right)}; \end{aligned}$$

consequently the proposed equation is,

$$\begin{aligned} &\left\{x^2-\frac{1}{2}(3-\sqrt{-35})x-2\right\}\times \\ &\left\{x^2-\frac{1}{2}(3+\sqrt{-35})x-2\right\}=0. \end{aligned}$$

[10.] The factors of $x^4-2x^3+5x^2-3x+7$ are,

$$\begin{aligned} &x^2\pm\sqrt{(2x^3-5x^2+3x-7)} \\ &=x^2-x\pm\sqrt{(-4x^2+3x-7)}. \end{aligned}$$

Here the expression under the radical is,

$$-(4x^2-3x+7);$$

and, since $4 \times 4 \times 7 > 3^2$, it follows that whatever real value be put for $x$, the irrational part of each of the factors above will be imaginary: consequently the product of those factors can never be zero for any real value of $x$; therefore all the roots of the proposed equation must be imaginary. (See Algebra, page 182.)

[11.] Multiplying by 5, in order that the leading term may be a square, the first member of the equation becomes

$$25x^4 - 180x^3 + 336x^2 - 43 \cdot 2x + 1 \cdot 44;$$

the factors of which are,

$$\begin{aligned} &5x^2 \pm \sqrt{(180x^3 - 336x^2 + 43 \cdot 2x - 1 \cdot 44)} \\ &= 5x^2 - 18x \pm \sqrt{(-12x^2 + 43 \cdot 2x - 1 \cdot 44)} \\ &= 5x^2 - 18x + 1 \cdot 2 \pm \sqrt{0}; \end{aligned}$$

consequently the proposed equation is, $(5x^2 - 18x + 1 \cdot 2)^2 = 0$; or, dividing by $5^2$, it is $(x^2 - 3 \cdot 6x + \cdot 24)^2 = 0$.

[12.] The factors of $x^4 + ax^3 + bx^2 + cx + \left(\frac{c}{a}\right)^2$ are,

$$\begin{aligned} &x^2 \pm \sqrt{\left(-ax^3 - bx^2 - cx - \frac{c^2}{a^2}\right)} \\ &= x^2 + \frac{a}{2}x \pm \sqrt{\left(\frac{a^2}{4}x^2 - b\right)x^2 - cx - \frac{c^2}{a^2}} \\ &= x^2 + \frac{a}{2}x + \frac{c}{a} \pm \sqrt{\left(\frac{a^2}{4} - b + \frac{2c}{a}\right)x^2}; \end{aligned}$$

consequently the factors are,

$$x^2 + \left\{\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b + \frac{2c}{a}\right)}\right\}x + \frac{c}{a};$$

and the product of these is the first member of the proposed equation.

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## PAGE 196, ALGEBRA.

[1.] The first member of this equation, when multiplied by $x$, is $x^4 - x^3 + 2x$, and the factors of this are found as follows:

$$x^2 \pm \sqrt{(x^3 - 2x)}$$
$$= x^2 - \frac{1}{2}x \pm \sqrt{\left(\frac{1}{4}x^2 - 2x\right)}$$
$$= x^2 - \frac{1}{2}x + 1 \pm \sqrt{\left(\frac{9}{4}x^2 - 3x + 1\right)}$$
$$= x^2 - \frac{1}{2}x + 1 \pm \sqrt{\left(\frac{3}{2}x - 1\right)^2};$$

consequently the factors are,

$$\left\{x^2 - \frac{1}{2}x + 1 + \left(\frac{3}{2}x - 1\right)\right\}\left\{x^2 - \frac{1}{2}x + 1 - \left(\frac{3}{2}x - 1\right)\right\};$$

that is, $(x^2 + x)(x^2 - 2x + 2)$;
therefore removing the factor $x$, introduced at the outset, the proposed equation is,

$$(x + 1)(x^2 - 2x + 2) = 0.$$

[2.] Multiplying by $x$, the first member becomes

$$x^4 - \frac{1}{2}x^3 + \frac{1}{4}x,$$

of which the factors are

$$x^2 \pm \sqrt{\left(\frac{1}{2}x^3 - \frac{1}{4}x\right)}$$
$$= x^2 - \frac{1}{4}x \pm \sqrt{\left(\frac{1}{16}x^2 - \frac{1}{4}x\right)}$$
$$= x^2 - \frac{1}{4}x + \frac{1}{4} \pm \sqrt{\left(\frac{9}{16} - \frac{3}{8}x + \frac{1}{16}\right)}$$
$$= x^2 - \frac{1}{4}x + \frac{1}{4} \pm \sqrt{\left(\frac{3}{4}x - \frac{1}{4}\right)^2};$$

consequently the factors are,

$$\left\{x^2 - \frac{1}{4}x + \frac{1}{4} + \left(\frac{3}{4}x - \frac{1}{4}\right)\right\}\left\{x^2 - \frac{1}{4}x + \frac{1}{4} - \left(\frac{3}{4}x - \frac{1}{4}\right)\right\};$$

that is, $\left(x^2 + \frac{1}{2}x\right)\left(x^2 - x + \frac{1}{2}\right)$;
therefore, dividing by the $x$ originally introduced, the proposed equation is

$$\left(x + \frac{1}{2}\right)\left(x^2 - x + \frac{1}{2}\right) = 0.$$

[3.] Multiplying the first member by $x$, it becomes $x^4 - ax^3 + 2a^3x$, the factors of which are,

$$x^2 \pm \sqrt{(ax^3 - 2a^3x)}$$

$$= x^2 - \frac{a}{2}x \pm \sqrt{\left(\frac{a^2}{4}x^2 - 2a^3x\right)}$$

$$= x^2 - \frac{a}{2}x + a^2 \pm \sqrt{\left(\frac{9}{4}a^2x^2 - 3a^3x + a^4\right)}$$

$$= x^2 - \frac{a}{2}x + a^2 \pm \sqrt{\left(\frac{3}{2}ax - a^2\right)^2};$$

consequently the factors are

$$\left\{x^2 - \frac{a}{2}x + a^2 + \left(\frac{3}{2}ax - a^2\right)\right\}\left\{x^2 - \frac{a}{2}x + a^2 - \left(\frac{3}{2}ax - a^2\right)\right\};$$

that is, $(x^2 + ax)(x^2 - 2ax + 2a^2)$;

therefore, dividing by the $x$, introduced at the commencement, the proposed equation is,

$$(x + a)(x^2 - 2ax + 2a^2) = 0.$$

[4.] Multiplying the first member by $x$, it becomes $x^4 - 3x^3 + 4x^2 - 12x$, the factors of which are,

$$x^2 \pm \sqrt{(3x^3 - 4x^2 + 12x)}$$

$$= x^2 - \frac{3}{2}x \pm \sqrt{\left(-\frac{7}{4}x^2 + 12x\right)}$$

$$= x^2 - \frac{3}{2}x + 2 \pm \sqrt{\left(\frac{9}{4}x^2 + 6x + 4\right)}$$

$$= x^2 - \frac{3}{2}x + 2 \pm \sqrt{\left(\frac{3}{2}x + 2\right)^2};$$

consequently the factors are,

$$\left\{x^2 - \frac{3}{2}x + 2 + \left(\frac{3}{2}x + 2\right)\right\}\left\{x^2 - \frac{3}{2}x + 2 - \left(\frac{3}{2}x + 2\right)\right\};$$

that is, $(x^2 + 4)(x^2 - 3x)$;

therefore dividing by the $x$, introduced at first, the proposed equation is $(x - 3)(x^2 + 4) = 0.$

[5.] Multiplying the first member by $x$, it becomes $x^4+7x^3+11x^2-4x$, the factors of which are,

$$x^2\pm\sqrt{(-7x^3-11x^2+4x)}$$
$$=x^2+\frac{7}{2}x\pm\sqrt{\left(\frac{5}{4}x^2+4x\right)}$$
$$=x^2+\frac{7}{2}x-\frac{1}{2}\pm\sqrt{\left(\frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{4}\right)}$$
$$=x^2+\frac{7}{2}x-\frac{1}{2}\pm\sqrt{\left(\frac{1}{2}x+\frac{1}{2}\right)^2};$$

consequently the factors are,

$$\left\{x^2+\frac{7}{2}x-\frac{1}{2}+\left(\frac{1}{2}x+\frac{1}{2}\right)\right\}\left\{x^2+\frac{7}{2}x-\frac{1}{2}-\left(\frac{1}{2}x+\frac{1}{2}\right)\right\};$$

that is, $(x^2+4x)(x^2+3x-1)$;

therefore, dividing by $x$, the factor introduced at first, the proposed equation is $(x+4)(x^2+3x-1)=0$.

[6.] Multiplying by $x$, the first member becomes $x^4-3x^3+5x^2-3x$, the factors of which are,

$$x^2\pm\sqrt{(3x^3-5x^2+3x)}$$
$$=x^2-\frac{3}{2}x\pm\sqrt{\left(-\frac{11}{4}x^2+3x\right)}$$
$$=x^2-\frac{3}{2}x+\frac{3}{2}\pm\sqrt{\left(\frac{1}{4}x^2-\frac{3}{2}x+\frac{9}{4}\right)}$$
$$=x^2-\frac{3}{2}x+\frac{3}{2}\pm\sqrt{\left(\frac{1}{2}x-\frac{3}{2}\right)^2};$$

consequently the factors are,

$$\left\{x^2-\frac{3}{2}x+\frac{3}{2}+\left(\frac{1}{2}x-\frac{3}{2}\right)\right\}\left\{x^2-\frac{3}{2}x+\frac{3}{2}-\left(\frac{1}{2}x-\frac{3}{2}\right)\right\};$$

that is, $(x^2-x)(x^2-2x+3)$;

therefore, dividing by $x$, the factor introduced at the commencement, the proposed equation is,

$$(x-1)(x^2-2x+3)=0.$$

[7.]

Multiplying by $x$, the first member becomes $x^4+3x^3+3x^2+2x$, the factors of which are,

$$x^2 \pm \surd(-3x^3-3x^2-2x)$$
$$=x^2+\frac{3}{2}x \pm \surd(-\frac{3}{4}x^2-2x)$$
$$=x^2+\frac{3}{2}x+\frac{1}{2} \pm \sqrt{\left(\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}\right)}$$
$$=x^2+\frac{3}{2}x+\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}x-\frac{1}{2}\right)^2};$$

consequently the factors are,

$$\left\{x^2+\frac{3}{2}x+\frac{1}{2}+\left(\frac{1}{2}x-\frac{1}{2}\right)\right\}\left\{x^2+\frac{3}{2}x+\frac{1}{2}-\left(\frac{1}{2}x-\frac{1}{2}\right)\right\};$$

that is, $(x^2+2x)(x^2+x+1)$;

therefore, dividing by $x$, the factor introduced at the beginning, the proposed equation is,

$$(x+2)(x^2+x+1)=0.$$

[8.]

Multiplying by $x$, the first member becomes $x^4+2x^3+3x^2+2x$, the factors of which are,

$$x^2 \pm \surd(-2x^3-3x^2-2x)$$
$$=x^2+x \pm \surd(-2x^2-2x)$$
$$=x^2+x+1 \pm \surd(1);$$

consequently the factors are,

$$(x^2+x+1-1)(x^2+x+1+1);$$

that is, $(x^2+x)(x^2+x+2)$;

therefore, dividing by $x$, the factor introduced at the commencement, the proposed equation is,

$$(x+1)(x^2+x+2)=0.$$

[9.]

Multiplying by $x$, the first member becomes $x^4-6x^3+10x^2-3x$, the factors of which are,

$$\begin{aligned} &x^2 \pm \sqrt{(6x^3 - 10x^2 + 3x)} \\ &= x^2 - 3x \pm \sqrt{(-x^2 + 3x)} \\ &= x^2 - 3x + \frac{1}{2} \pm \sqrt{\left(\frac{1}{4}\right)}; \end{aligned}$$

consequently the factors are,

$$\left(x^2 - 3x + \frac{1}{2} - \frac{1}{2}\right)\left(x^2 - 3x + \frac{1}{2} + \frac{1}{2}\right);$$

$$\text{that is, } (x^2 - 3x)(x^2 - 3x + 1);$$

therefore, dividing by $x$, the factor introduced above, the proposed equation is $(x - 3)(x^2 - 3x + 1) = 0$.

[10.] Multiplying by $2x$, the first member becomes $4x^4 + 16x^3 + 18x^2 + 18x$, the factors of which are,

$$\begin{aligned} &2x^2 \pm \sqrt{(-16x^3 - 18x^2 - 18x)} \\ &= 2x^2 + 4x \pm \sqrt{(-2x^2 - 18x)} \\ &= 2x^2 + 4x + \frac{3}{2} \pm \sqrt{\left(4x^2 - 6x + \frac{9}{4}\right)} \\ &= 2x^2 + 4x + \frac{3}{2} \pm \sqrt{\left(2x - \frac{3}{2}\right)^2}; \end{aligned}$$

consequently the factors are,

$$\left\{2x^2 + 4x + \frac{3}{2} + \left(2x - \frac{3}{2}\right)\right\}\left\{2x^2 + 4x + \frac{3}{2} - \left(2x - \frac{3}{2}\right)\right\};$$

$$\text{that is, } (2x^2 + 6x)(2x^2 + 2x + 3);$$

therefore, dividing by $2x$, the factor introduced at the beginning, the proposed equation is,

$$(x + 3)(2x^2 + 2x + 3) = 0.$$

[11.] Multiplying by $x$, the first member becomes $4x^4 + 16x^3 + 23x^2 + 20x = 0$, the factors of which are,

$$\begin{aligned} &2x^2 \pm \sqrt{(-16x^3 - 23x^2 - 20x)} \\ &= 2x^2 + 4x \pm \sqrt{(-7x^2 - 20x)} \\ &= 2x^2 + 4x + 2 \pm \sqrt{(x^2 - 4x + 4)} \\ &= 2x^2 + 4x + 2 \pm \sqrt{(x - 2)^2}; \end{aligned}$$

consequently the factors are,

$$\{2x^2+4x+2+(x-2)\}\{2x^2+4x+2-(x-2)\};$$

$$\text{that is, } (2x^2+5x)(2x^2+3x+4);$$

therefore, dividing by $x$, the factor introduced at the outset, the proposed equation is $(2x+5)(2x^2+3x+4)=0$.

[12.] Multiplying by $x$, the first member is $x^4-px+qx^2-pqx$, the factors of which are,

$$x^2\pm\sqrt{(px^3-qx^2+pqx)}$$

$$=x^2-\frac{p}{2}x\pm\sqrt{\left(\frac{p^2}{4}x^2-qx^2+pqx\right)}$$

$$=x^2-\frac{p}{2}x+\frac{q}{2}\pm\sqrt{\left(\frac{p^2}{4}x^2+\frac{pq}{2}x+\frac{q^2}{4}\right)}$$

$$=x^2-\frac{p}{2}x+\frac{q}{2}\pm\sqrt{\left(\frac{p}{2}x+\frac{q}{2}\right)^2};$$

consequently the factors are,

$$\left\{x^2-\frac{p}{2}x+\frac{q}{2}-\left(\frac{p}{2}x+\frac{q}{2}\right)\right\}\left\{x^2-\frac{p}{2}x+\frac{q}{2}+\left(\frac{p}{2}x+\frac{q}{2}\right)\right\};$$

$$\text{that is, } (x^2-px)(x^2+q);$$

therefore, dividing by $x$, the factor introduced at the beginning, the proposed equation is,

$$(x-p)(x^2+q)=0.$$

For some further particulars in reference to this method of decomposing polynomials into their component factors, the student is directed to Note (B), at the end of the Key.

## TRANSFORMATION OF EQUATIONS.

### PAGE 204.

$$\begin{array}{lrrrrl} \text{[1.]} & 1 & -7 & 4 & -8 & (1 \\ & & 1 & -6 & -2 & \\ \cline{3-5} & & -6 & -2 & -10 & \\ & & 1 & -5 & & \\ \cline{3-4} & & -5 & -7 & & \\ & & 1 & & & \\ \cline{3-3} & & -4 & & & \end{array}$$

therefore the transformed equation is $x^3-4x^2-7x-10=0$, the roots of which are severally equal to those of the proposed equation diminished by 1.

$$\begin{array}{lrrrrrl} \text{[2.]} & 2 & 0 & -13 & 10 & -19 & (1 \\ & & 2 & 2 & -11 & -1 & \\ \cline{3-6} & & 2 & -11 & -1 & -20 & \\ & & 2 & 4 & -7 & & \\ \cline{3-5} & & 4 & -7 & -8 & & \\ & & 2 & 6 & & & \\ \cline{3-4} & & 6 & -1 & & & \\ & & 2 & & & & \\ \cline{3-3} & & 8 & & & & \end{array}$$

hence the transformed equation is $2x^4+8x^3-x^2-8x-20=0$.

$$\begin{array}{lrrrrl} \text{[3.]} & 3 & 2 & -6 & 5 & (-2 \\ & & -6 & 8 & -4 & \\ \cline{3-5} & & -4 & 2 & 1 & \\ & & -6 & 20 & & \\ \cline{3-4} & & -10 & 22 & & \\ & & -6 & & & \\ \cline{3-3} & & -16 & & & \end{array}$$

hence the transformed equation is $3x^3-16x^2+22x+1=0$.

[4.]

$$\begin{array}{rrrrl}
4 & -3 & 2 & -1 & -10\,(-3 \\
 & -12 & 45 & -141 & 426 \\
 & \overline{-15} & \overline{47} & \overline{-142} & \overline{416} \\
 & -12 & 81 & -384 & \\
 & \overline{-27} & \overline{128} & \overline{-526} & \\
 & -12 & 117 & & \\
 & \overline{-39} & \overline{245} & & \\
 & -12 & & & \\
 & \overline{-51} & & &
\end{array}$$

consequently the transformed equation is,

$$4x^4-51x^3+245x^2-526x=-416.$$

[5.]

$$\begin{array}{rrrl}
5 & 0 & -6 & 12\,(5 \\
 & 25 & 125 & 595 \\
 & \overline{25} & \overline{119} & \overline{607} \\
 & 25 & 250 & \\
 & \overline{50} & \overline{369} & \\
 & 25 & & \\
 & \overline{75} & &
\end{array}$$

therefore the transformed equation is,

$$5x^3+75x^2+369x+607=0.$$

[6.]

$$\begin{array}{rrrrl}
3 & 0 & 1 & -5 & 8\,(4 \\
 & 12 & 48 & 196 & 764 \\
 & \overline{12} & \overline{49} & \overline{191} & \overline{772} \\
 & 12 & 96 & 580 & \\
 & \overline{24} & \overline{145} & \overline{771} & \\
 & 12 & 144 & & \\
 & \overline{36} & \overline{289} & & \\
 & 12 & & & \\
 & \overline{48} & & &
\end{array}$$

hence the transformed equation is,

$$3x^4+48x^3+289x^2+771x+772=0.$$

$$\begin{array}{lrrrl}
\text{[7.]}\ 1 & 6 & 5 & 1 & (-2 \\
 & \underline{-2} & \underline{-8} & \underline{6} & \\
 & 4 & -3 & 7 & \\
 & \underline{-2} & \underline{-4} & & \\
 & 2 & -7 & & \\
 & \underline{-2} & & & \\
 & 0 & & &
\end{array}$$

consequently the transformed equation is $x^3 - 7x + 7 = 0$.

$$\begin{array}{lrrrrl}
\text{[8.]}\ 19 & -22 & -35 & -16 & -2 & (3 \\
 & \underline{57} & \underline{105} & \underline{210} & \underline{582} & \\
 & 35 & 70 & 194 & 580 & \\
 & \underline{57} & \underline{276} & \underline{1038} & & \\
 & 92 & 346 & 1232 & & \\
 & \underline{57} & \underline{447} & & & \\
 & 149 & 793 & & & \\
 & \underline{57} & & & & \\
 & 206 & & & &
\end{array}$$

therefore the transformed equation is,

$$19x^4 + 206x^3 + 793x^2 + 1232x + 580 = 0.$$

$$\begin{array}{lrrrl}
\text{[9.]}\ 12 & 24 & -58 & 25 & (\cdot 7 \\
 & \underline{8\cdot 4} & \underline{22\cdot 68} & \underline{-24\cdot 724} & \\
 & 32\cdot 4 & -35\cdot 32 & \cdot 276 & \\
 & \underline{8\cdot 4} & \underline{28\cdot 56} & & \\
 & 40\cdot 8 & -6\cdot 76 & & \\
 & \underline{8\cdot 4} & & & \\
 & 49\cdot 2 & & &
\end{array}$$

consequently the transformed equation is,

$$12x^3 + 49\cdot 2x^2 - 6\cdot 76x + \cdot 276 = 0.$$

$$\begin{array}{lrrrl}
\text{[10.]}\ 12 & 49\cdot 2 & -6\cdot 76 & \cdot 276 & (\cdot 06 \\
 & \underline{\cdot 72} & \underline{2\cdot 9952} & \underline{-\cdot 225888} & \\
 & 49\cdot 92 & -3\cdot 7648 & \cdot 050112 & \\
 & \underline{\cdot 72} & \underline{3\cdot 0384} & & \\
 & 50\cdot 64 & -\cdot 7264 & & \\
 & \underline{\cdot 72} & & & \\
 & 51\cdot 36 & & &
\end{array}$$

consequently the transformed equation is,

$$12x^3 + 51\cdot36x^2 - \cdot7264x + \cdot050112 = 0.$$

It thus appears from the last two examples, that the equation whose roots are those of the equation

$$12x^3 + 24x^2 - 58x + 25 = 0,$$

when each of them is diminished by ·76, is

$$12x^3 + 51\cdot36x^2 - \cdot7264x + \cdot050112 = 0.$$

And by diminishing the roots in this way, figure after figure, having regard to the local value of that figure, we may diminish the roots of an equation by any number we please, by a series of uniform steps. The following examples on the numerical solution of equations, merely exhibits these successive steps united together in one continuous system of operations.

## SOLUTION OF NUMERICAL EQUATIONS.

### PAGE 218.

[1.] The factors of the first member of the proposed equation are,

$$x \pm \surd(-x^4 + 8x + 15)$$
$$= x - 4 \pm \surd(-x^4 + 31);$$

the $-x^4$ being put under the radical sign rather than $-x^2$, as we shall thus more readily discover the smallest positive number for which the whole expression under the radical becomes negative. This smallest number is evidently 3, and the expression is negative also for all greater numbers; so that the first figure of the root cannot be a number greater than 2. Upon diminishing the roots by 2, we find that this number is not too great; we therefore continue the operation as follows:

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 1 | −8 | 15 ( 2·30277563 . . . . |
| | 2 | 4 | 10 | 4 |
| | 2 | 5 | 2 | 11 (1) |
| | 2 | 8 | 26 | 10·8741 |
| | 4 | 13 | 28 (1) | ·1259 (2) |
| | 2 | 12 | 8·247 | 906670 |
| | 6 | 25 (1) | 36·247 | 352330 (3) |
| | 2 | 2·49 | 9·021 | 317955 |
| | 8 (1) | 27·49 | 45·268 (2) | 34375 (4) |
| | ·3 | 2·58 | 6552 | 31813 |
| | 8·3 | 30·07 | 45·3335,2 | 2562 (5) |
| | ·3 | 2·67 | 656 | 2273 |
| | 8·6 | 32·74 (2) | 45·3991 (3) | 289 |
| | ·3 | 18 | 230 | 273 |
| | 8·9 | 32·75,8 | 45·422,1 | 16 |
| | ·3 | 2 | 23 | 14 |
| | ,9·,2 (2) | 32·7,8 | 45·445 (4) | 2 |
| | | 2 | 2 | |
| | | ,3,2·,8 (3) | 45·44,7 | |
| | | | 2 | |
| | | | 4,5·,4,5 (5) | |

consequently, the positive root of the proposed equation is

$$x = 2\cdot30277563.$$

[2.] It is obvious that if 0 be put for $x$ in this equation, that the first member becomes negative, namely $-1$; and that if 1 be substituted for $x$, the first member becomes positive, namely 1: we may therefore conclude (Algebra, page 207, art. 93) that a positive root lies between 0 and 1, and consequently that the first significant figure is a decimal. But, by applying the method of decomposition into factors, we shall get information still more definite. The factors of the first member, after multiplication by $x$, are

$$x^2 \pm \sqrt{(5x^3 - 7x^2 + x)}$$
$$= x^2 - \frac{5}{2}x \pm \sqrt{\left(-\frac{3}{4}x^2 + x\right)}$$
$$= x^2 - \frac{5}{2}x + \frac{3}{8} \pm \sqrt{\left(-\frac{7}{8}x + \frac{9}{64}\right)}.$$

From the expression under the radical sign, we infer that the proposed equation cannot have any positive root so great as 1, because that expression becomes negative not only for $x = 1$, but for $x$ equal to any greater number. Further, the expression is negative for $x = \frac{1}{6}$, and for all greater values; and as $\frac{1}{6} = {\cdot}16\ldots$, it will be proper to try ·1 for the first figure of the root; which is found to succeed, as $x = {\cdot}2$ is shown above to give a negative result for the absolute term of the transformed equation:* the operation is therefore as follows:

| | | | |
|---|---|---|---|
| 1 | −5 | ·7 | 1 (·160713 . . . |
| | ·1 | −·49 | ·651 |
| | −4·9 | 6·51 | ·349 (1) |
| | ·1 | −·48 | ·345096 |
| | −4·8 | 6·03 (1) | 3904 (2) |
| | ·1 | −·2784 | 3832 |
| | −4·7 (1) | 5·7516 | ·72 (3) |
| | 6 | −·2748 | 55 |
| | −4·64 | 5·4768 (2) | 17 |
| | 6 | −32 | 16 |
| | −4·58 | 5·4736 | 1 |
| | 6 | −32 | |
| | −4·52 (2) | 5·47 (3) | |

consequently the only positive root the equation has, is $x = {\cdot}160713$.

---

* When a number which exceeds the greatest positive root of an equation is substituted for $x$ in the first member of that equation, the result is always *positive*, it being understood that zero only forms the second member of the equation. The result here spoken of is no other than the final or absolute term of the transformed equation, which arises from diminishing the roots of the proposed by the number thus substituted for $x$. But when the absolute term of the equation is transposed, and therefore, with changed sign, forms the second member of the equation, as in the above arrangement for the development of the root, then the absolute term in each transformed equation becomes in like manner transposed and changed in sign: hence, under this arrangement, a number exceeding the greatest positive root of the equation will always cause the absolute term in the transformed equation to be *minus;* and, in trying lower numbers, we may be certain that a root is passed over as soon as the absolute term of the corresponding transformed equation becomes *plus*.

[3.] A slight inspection of this equation is sufficient to show that the first member becomes *minus* for $x=0$, and *plus* for $x=1$: therefore, by the principle of art. 93, page 207, a positive root lies between 0 and 1; that is, the first figure of the root is a decimal. But this is information too indefinite to be of much avail in enabling us to fix upon the leading figure of the root, and thus to commence the operation for its complete development: we shall therefore seek for more precise directions for finding this leading figure, from an examination of the factors of the first member of the equation. These factors are,

$$x \pm \surd(-x^3 - x + 1)$$
$$= x - 1 \pm \surd(-x^3 - 3x + 2).$$

Now, it is evident that the expression under the radical sign becomes negative for $x = \cdot 6$, and for every greater number. Our course is thus obvious: we are to diminish the roots by $\cdot 5$, and then, if need be, by $\cdot 4$, and so on till we reach a number giving the same sign to the absolute term of the transformed equation that the given absolute term is affected with; that is, the sign *plus:* this number will be the leading figure of the root; it proves to be 5, and the operation is as follows:

$$\begin{array}{rrrr}
1 & 1 & 1 & 1\ (\cdot 543689 \\
 & \cdot 5 & \cdot 75 & \cdot 875 \\
 & \overline{1\cdot 5} & \overline{1\cdot 75} & \overline{\cdot 125}^{(1)} \\
 & \cdot 5 & 1 & \cdot 114064 \\
 & \overline{2} & \overline{2\cdot 75}^{(1)} & \overline{10936}^{(2)} \\
 & \cdot 5 & \cdot 1016 & 8888 \\
 & \overline{2\cdot 5}^{(1)} & \overline{2\cdot 8516} & \overline{2048}^{(3)} \\
 & 4 & \cdot 1032 & 1785 \\
 & \overline{2\cdot 54} & \overline{2\cdot 9548}^{(2)} & \overline{263}^{(4)} \\
 & 4 & 79 & 238 \\
 & \overline{2\cdot 58} & \overline{2\cdot 962{,}7} & \overline{25} \\
 & 4 & 79 & \underline{26} \\
 & \overline{2\cdot{,}6{,}2}^{(2)} & \overline{2\cdot 971}^{(3)} & \\
 & & 2 & \\
 & & \overline{2\cdot 973} & \\
 & & 2 & \\
 & & \overline{2\cdot{,}9{,}7{,}5}^{(4)} & 
\end{array}$$

consequently the root is $x = \cdot 543689$.

[4.] Multiplying by $2x$, and decomposing, the factors are,

$$2x^2 \pm \sqrt{(6x^3 - 12x^2 + 16x)}$$
$$= 2x^2 - \frac{3}{2}x \pm \sqrt{(-\frac{39}{4}x^2 + 16x)}.$$

In considering the sign of the expression under the radical, we may evidently discard the factor $x$ which enters both terms, and pay attention only to $-\frac{39}{4}x + 16$, which obviously becomes negative for $x = 2$, and for every greater number: hence we commence the operation by diminishing the roots by 1, and proceed as follows:

| | | | | |
|---|---|---|---|---|
| 2 | −3 | 0 | 6 | 8 (1·414213562 .... |
| | 2 | −1 | −1 | 5 |
| | −1 | −1 | 5 | 3 (1) |
| | 2 | 1 | 0 | 2·8512 |
| | 1 | 0 | 5 (1) | ·1488 (2) |
| | 2 | 3 | 2·128 | ·10422022 |
| | 3 | 3 (1) | 7·128 | 4457978 (3) |
| | 2 | 2·32 | 3·184 | 4231068 |
| | 5 (1) | 5·32 | 10·312 (2) | 226910 (4) |
| | ·8 | 2·64 | ·110022 | 212497 |
| | 5·8 | 7·96 | 10·422022 | 14413 (5) |
| | ·8 | 2·96 | ·110846 | 10627 |
| | 6·6 | 10·92 (2) | 10·532868 (3) | 3786 |
| | ·8 | 822 | 44802 | 3188 |
| | 7·4 | 11·0022 | 10·577670 | 598 |
| | ·8 | 824 | 4494 | 531 |
| | 8·2 (2) | 11·0846 | 10·62261 (4) | 67 |
| | 2 | 826 | 225 | 64 |
| | 8·22 | 11·1672 (3) | 10·62486 | 3 |
| | 2 | 332 | 225 | 2 |
| | 8·24 | 11·2004 | 10·6271 (5) | 1 |
| | 2 | 332 | 1 | |
| | 8·26 | 11·234 | 10·6272 | |
| | 2 | 33 | | |
| | 8·28 (3) | 11·267 (4) | | |
| | 8 | 2 | | |
| | 8·29 | 11·27 | | |

consequently the root, as far as nine places of decimals, is $x = 1·414213562$; or, to the nearest decimal in the sixth place, it is $x = 1·414214$.

[5.] For the factors of the first member of this equation we have,

$$x^2 \pm \sqrt{(-5x^3 + 3x^2 - 15x + 25)}$$
$$= x^2 - \frac{3}{2} \pm \sqrt{\left(-5x^3 - 15x + 27\frac{1}{4}\right)}.$$

The expression under the radical sign is obviously negative for $x = 2$; it is also negative even for $x = 1\frac{1}{2}$, as well as for all greater numbers. We therefore take 1 for the first figure of the root, and operate as follows:

| | | | | |
|---|---|---|---|---|
| 1 | 5 | −3 | 15 | 25 (1·21652 |
| | 1 | 6 | 3 | −18 |
| | 6 | 3 | 18 | −7 (1) |
| | 1 | 7 | 10 | 6·3936 |
| | 7 | 10 | 28 (1) | ·6064 (2) |
| | 1 | 8 | 3·968 | ·3655 |
| | 8 | 18 (1) | 31·968 | ·2409 (3) |
| | 1 | 1·84 | 4·344 | ·2216 |
| | 9 (1) | 19·84 | 36·312 (2) | 193 (4) |
| | ·2 | 1·88 | ·237 | 186 |
| | 9·2 | 21·72 | 36·54,9 | 7 |
| | ·2 | 1·92 | ·24 | 7 |
| | 9·4 | 23·64 (2) | 36·79 (3) | |
| | ·2 | ·10 | ·14 | |
| | 9·6 | 23·7,4 | 36·9,3 | |
| | ·2 | ·1 | ·14 | |
| | 9·8 (2) | 23·8 | 3,7·1 (4) | |
| | 1 | ·1 | | |
| | 9·,81 | 2,4 (3) | | |

consequently the root is $x = 1·21652$, which is true as far as five places of decimals. It may here be observed that in equations of the fourth degree, our curtailments to prevent an unnecessary accumulation of decimals may in general begin after the first decimal in the root is found, provided there be but one place of integers in the root. If we confine the decimals in the last column of work to the limit reached at this second step, we shall usually obtain the root as far as six decimals. The reason that only five is furnished above is, that at the last step but one, namely, that due to the 5, the remainder is only a single figure 7, instead of two figures as might have been expected.

[6.] Multiplying by $x$, the equation is $x^4 + x^2 - 13x = 0$; and for the factors of the first member, we have

$$x \pm \sqrt{(-x^4 + 13x)}.$$

Discarding the factor $x$, we have to examine the sign of $-x^3 + 13$, which is obviously negative for $x = 3$, and for all greater numbers: we commence, therefore, by diminishing the roots by 2.

$$\begin{array}{rrrl}
1 & 0 & 1 & 13\,(2\cdot209753 \\
 & 2 & 4 & 10 \\
 & \overline{2} & \overline{5} & \overline{3}^{(1)} \\
 & 2 & 8 & 2\cdot848 \\
 & \overline{4} & \overline{13}^{(1)} & \overline{\cdot152}^{(2)} \\
 & 2 & 1\cdot24 & \cdot14022 \\
 & \overline{6}^{(1)} & \overline{14\cdot24} & \overline{1178}^{(3)} \\
 & \cdot2 & 1\cdot28 & 1095 \\
 & \overline{6\cdot2} & \overline{15\cdot52}^{(2)} & \overline{83} \\
 & \cdot2 & 595 & 78 \\
 & \overline{6\cdot4} & \overline{15\cdot57,95} & \overline{5} \\
 & \cdot2 & 6 & 5 \\
 & \overline{6\cdot6}^{(2)} & \overline{15\cdot64}^{(3)} & \overline{\phantom{5}} \\
 & 09 & & \\
 & \overline{6\cdot61} & & 
\end{array}$$

consequently the root is $x = 2\cdot209753$.

[7.] This is merely to find the cube root of 926859375, which by the general method is as follows:

$$\begin{array}{rrrl}
1' & 0 & 0 & 926,859,375\ (975 \\
 & 9 & 81 & 729 \\
 & \overline{9} & \overline{81} & \overline{197859}^{(1)} \\
 & 9 & 162 & 183673 \\
 & \overline{18} & \overline{243}^{(1)} & \overline{14186375}^{(2)} \\
 & 9 & 1939 & \underline{14186375} \\
 & \overline{27}^{(1)} & \overline{26239} & \\
 & 7 & 1988 & \\
 & \overline{277} & \overline{28227}^{(2)} & \\
 & 7 & 14575 & \\
 & \overline{284} & \underline{\overline{2837275}} & \\
 & 7 & & \\
 & \overline{291}^{(2)} & & \\
 & 5 & & \\
 & \underline{\overline{2915}} & &
\end{array}$$

consequently the cube root of the proposed number is 975.

[8.] Dividing by 2, the first member of the equation is $x^4-8x^3+20x^2-15x+\cdot 5$; and decomposing this into factors, we have

$$\begin{aligned}
& x^2 \pm \sqrt{(8x^3-20x^2+15x-\cdot 5)} \\
&= x^2-4x\pm\sqrt{(-4x^2+15x-\cdot 5)} \\
&= x^2-4x+2\pm\sqrt{(-x+3\cdot 5)}.
\end{aligned}$$

It follows from this that there can be no root of the equation greater than $3\cdot 5$; that is to say, all its real roots must lie between 0 and $3\cdot 5$. We may commence, therefore, by diminishing the roots by 3, and then by 2, and so on, if need be, till we arrive at a transformed equation in which the absolute term is *plus*. The number to which this transformation is due will be the leading figure of a root.

$$\begin{array}{rrrrrl}
1 & -8 & 20 & -15 & -\cdot 5 & (\,3 \\
 & 3 & -15 & 15 & 0 & \\
\hline
 & -5 & 5 & 0 & -5 &
\end{array}$$

As the sign of the absolute number is *minus*, we next try 2.

$$\begin{array}{rrrrrl}
1 & -8 & 20 & -15 & -\cdot 5 & (\,2 \\
 & 2 & -12 & 16 & 2 & \\
\hline
 & -6 & 8 & 1 & -2\cdot 5 &
\end{array}$$

The sign being still *minus*, we proceed to try 1, which we see produces the looked-for change of sign, and we then continue the operation as follows:—

| | | | | |
|---|---|---|---|---|
| 1 | −8 | 20 | −15 | −·5 ( 1·284724 |
| | 1 | −7 | 13 | −2 |
| | −7 | 13 | −2 | 1·5 (1) |
| | 1 | −6 | 7 | 1·0496 |
| | −6 | 7 | 5 (1) | ·4504 (2) |
| | 1 | −5 | ·248 | ·42554 |
| | −5 | 2 (1) | 5·248 | 2486 (3) |
| | 1 | −·76 | ·104 | 2105 |
| | −4 (1) | 1·24 | 5·352 (2) | 381 |
| | ·2 | −·72 | − 327 | 368 |
| | −3·8 | ·52 | 5·3193 | 13 |
| | ·2 | −·68 | − 52 | 11 |
| | −3·6 | −·16 (2) | 5·267 (3) | 2 |
| | ·2 | −·249 | − 4 | 2 |
| | −3·4 | −·409 | 5·263 | |
| | ·2 | −·243 | | |
| | −3·2 (2) | −·652 | | |
| | 8 | −·237 | | |
| | −3·12 | −·889 (3) | | |
| | 8 | − 12 | | |
| | −3·04 | −·901 | | |
| | 8 | | | |
| | −2·96 | | | |
| | 8 | | | |
| | −2·9 (3) | | | |

consequently a positive root of the proposed equation is $x = 1·284724$.

[9.] This equation, when rendered complete, is

$$x^4 + 0x^3 - 12x^2 + 12x - 3 = 0;$$

and as negative roots are converted into positive, and positive into negative, by changing the alternate signs of the complete equation, commencing these changes at the second term (Algebra, p. 211, foot-note), we shall have to find a positive root of the equation.*

---

* If the learner go forward to the solution of the equation [16],

$$x^4 - 0x^3 - 12x^2 - 12x - 3 = 0 \ldots (A).$$

Decomposing the first member, we have the factors

$$x^2 \pm \sqrt{(12x^2 + 12x + 3)};$$

and as, in the expression under the radical, four times the product of the extreme terms is equal to the square of the middle term, that expression is a square; hence the proposed equation ($A$) is

$$\{x^2 - (2\sqrt{3}.x + \sqrt{3})\}\{x^2 + (2\sqrt{3}.x + \sqrt{3})\} = 0;$$

so that all the roots would be obtained by solving the two equations,

$$x^2 - 2\sqrt{3}.x = \sqrt{3} \ldots (B)$$
$$x^2 + 2\sqrt{3}.x = -\sqrt{3}.$$

In the first of these the two roots are, one positive and the other negative, and in the second, both roots are negative (Algebra, p. 180); so that by extracting the positive root of the former equation, we shall obtain the value of $x$ sought. But as it may be instructive to the learner to see the process of solution applied directly to the equation ($A$), as well as to ($B$), we shall develope the positive root from each.

Resuming the factors $x^2 \pm \sqrt{(12x^2 + 12x + 3)}$, which in their present form give no information as to the situation of the sought root, we shall deduce from them a new pair of factors, with the view of getting a term involving $x^2$, with a negative sign, under the radical,

$$x^2 - 10 \pm \sqrt{(-8x^2 + 12x + 103)}.$$

It is easy to see here that for $x = 5$, and for all greater values, the expression under the radical becomes negative. If we were to employ 4, we should find the absolute number in the transformed equation to be negative; the next trial with 3 would succeed; so that the first figure of the root can be found with but little additional trouble. But, in order to show the application of a useful principle mentioned in the note at the end of the Algebra (p. 247), we shall proceed to find the leading figure in a different manner. From the irrational expression above, it appears that that expression is

page 175, and compare the operation with that at page 215 of the Algebra, attending to the remarks subjoined to the work at page 176, he will see that, by changing the alternate signs in any equation, we merely convert the positive roots of that equation into negative, and the negative roots into positive.

rendered negative for 4, and a fraction. Let 4 be put for $x$ in the rational expression which precedes the radical, the result is 6: introducing this with changed sign into the rational part, the factors become

$$x^2 - 16 \pm \sqrt{(-20x^2 + 12x + 16^2 + 3)};$$

from which it appears that the expression under the radical becomes negative for $x = 4$, and for all higher values of $x$: we may therefore commence with 3, and proceed as follows:

| | | | | |
|---|---|---|---|---|
| 1 | 0 | −12 | −12 | 3 ( 3·907378.... |
| | 3 | 9 | −9 | −63 |
| | 3 | −3 | −21 | 66 (1) |
| | 3 | 18 | 45 | 65·0241 |
| | 6 | 15 | 24 (1) | ·9759 (2) |
| | 3 | 27 | 48·249 | ·925621 |
| | 9 | 42 (1) | 72·249 | 50279 (3) |
| | 3 | 11·61 | 59·427 | 39844 |
| | 12 (1) | 53·61 | 131·676 (2) | 10435 (4) |
| | ·9 | 12·42 | ·55558 | 9299 |
| | 12·9 | 66·03 | 132·23158 | 1136 |
| | ·9 | 13·23 | ·556 | 1063 |
| | 13·8 | 79·26 (2) | 132·788 (3) | 73 |
| | ·9 | ·1092 | 24 | |
| | 14·7 | 79·3692 | 132·812 | |
| | ·9 | 11 | 24 | |
| | 15·6 (2) | 79·48 | 132·84 (4) | |
| | ·07 | | | |
| | 15·607 | | | |

Returning now to the quadratic equation ($B$), and replacing $\sqrt{3}$ by its value 1·73205081, we have

$$x^2 - 3{\cdot}46410162x = 1{\cdot}73205081;$$

∴ Rule II. page 178, Algebra,

$$2x - 3{\cdot}46410162 = \sqrt{(6{\cdot}92820324 + 12)}$$
$$= \pm 4{\cdot}3506554 \ldots.$$

$$\begin{array}{r} 3{\cdot}4641016\ldots. \\ 4{\cdot}3506554\ldots. \\ \hline 2)7{\cdot}814757\ldots.. \\ \hline \end{array}$$

$$\therefore x = 3{\cdot}907378\ldots..$$

This value of the root agrees with that determined above. The seventh decimal here would be a 5, and it is plain that the seventh decimal, supplied by the dividend 73 and divisor 13‸2, in the former operation, is also a 5; the two processes, therefore, confirm one another in the most satisfactory manner. The value of $x$ just found, with the minus prefixed, is the negative root required to be developed: to the nearest decimal in the sixth place it is $-3{\cdot}907379$.

In the second of the above methods of developing the root, we have followed the special rule for quadratic equations, which rule, however, will usually be found to involve more trouble than the general method adopted in the foregoing examples, when the known coefficients are very large numbers. The inconvenience of such numbers was less felt in the case above, because we knew that the square of the coefficient of $x$ was the square of $2\sqrt{3}$, so that we got 12 for this square without any work. We shall now, however, solve the same quadratic by the general method, as follows:

| | | |
|---|---|---|
| 1 | −3·46410162 | 1·73205081 ( 3·90737855 .... |
| | 3 | −1·39230486 |
| | −·46410162 | 3·12435567 (1) |
| | 3 | 3·09230854 |
| | 2·53589838 (1) | 3204713 (2) |
| | ·9 | 3040029 |
| | 3·4358983‸8 | 164684 (3) |
| | ·9 | 130506 |
| | 4·3358984 (2) | 34178 (4) |
| | 7 | 30454 |
| | 4·34289‸84 | 3724 |
| | 7 | 3480 |
| | 4·34990 (3) | 244 |
| | 3 | 218 |
| | 4·3502‸0 | 26 |
| | 3 | 22 |
| | 4·3505 (4) | 4 |
| | 7 | |
| | 4·3‸5‸0‸6 | |

The two additional decimals 55, here found, might evidently have also been added to the six decimals found by the first operation, as in the margin. The accuracy, therefore, of these three modes of operation is thus completely confirmed.

132)73(55
66
7
7

[10.] This equation, in a complete form, is

$$x^3+0x^2-3x+6=0;$$

therefore, changing alternate signs, we have to find a positive root of

$$x^3-0x^2-3x-6=0;$$

that is, of

$$x^3-3x-6=0\ldots(A);$$

multiplying by $x$, and decomposing the first member, we have the factors

$$x^2\pm\sqrt{(3x^2+6x)}$$
$$=x^2-4\pm\sqrt{(-5x^2+6x+16)};$$

consequently the equation ($A$) has no root so great as 3: we therefore commence the operation with 2 for the first figure of the root, and proceed as follows:

| | | | |
|---|---|---|---|
| 1 | 0 | −3 | 6 (2·3553014 |
| | 2 | 4 | 2 |
| | 2 | 1 | 4 (1) |
| | 2 | 8 | 3·267 |
| | 4 | 9 (1) | ·733 (2) |
| | 2 | 1·89 | ·660875 |
| | 6 (1) | 10·89 | 72125 (3) |
| | ·3 | 1·98 | 680139 |
| | 6·3 | 12·87 (2) | 41111 (4) |
| | ·3 | ·3475 | 40921 |
| | 6·6 | 13·2175 | 190 (5) |
| | ·3 | 3500 | 136 |
| | 6·9 (2) | 13·5675 (3) | 54 |
| | 5 | 3528 | 55 |
| | 6·95 | 13·60278 | |
| | 5 | 353 | |
| | 7·00 | 13·6381 (4) | |
| | 5 | 21 | |
| | 7·05 (3) | 13·6402 | |
| | 5 | 2 | |
| | 7·055 | 13·642 (5) | |
| | 5 | | |
| | 7·060 | | |

consequently the negative root of the proposed equation is $x = -2\cdot3553014$, which may be affirmed to be true to the nearest decimal in the seventh place.

[11.] This equation, with the alternate signs changed, is

$$x^3 + 5x^2 + 3x - 48 = 0;$$

and multiplying by $x$, we have for the factors of the first member, $x^2 \pm \sqrt{(-5x^3 - 3x^2 + 48x)}$;

and discarding the factor $x$ under the radical, it is plain that $-5x^2 - 3x + 48$ is negative for $x = 3$, and for all greater values: we therefore commence with 2 for the first figure of the root, and proceed with the development as follows:

$$\begin{array}{cccl}
1 & 5 & 3 & 48\,(2\cdot35833 \\
 & 2 & 14 & 34 \\
 & \overline{7} & \overline{17} & \overline{14} \\
 & 2 & 18 & 11\cdot517 \\
 & \overline{9} & \overline{35} & \overline{2\cdot483} \\
 & 2 & 3\cdot39 & 2\cdot1234 \\
 & \overline{11} & \overline{38\cdot39} & \overline{\cdot3596} \\
 & \cdot3 & 3\cdot48 & \cdot3454 \\
 & \overline{11\cdot3} & \overline{41\cdot87} & \overline{142} \\
 & \cdot3 & \cdot598 & 130 \\
 & \overline{11\cdot6} & \overline{42\cdot468} & \overline{12} \\
 & \cdot3 & \cdot60 & \underline{13} \\
 & \overline{11\cdot9} & \overline{43\cdot07} & \\
 & 5 & \cdot10 & \\
 & \overline{11\cdot95} & \overline{43\cdot17} & \\
 & 5 & \cdot1 & \\
 & \overline{12\cdot0} & \overline{43\cdot3} &
\end{array}$$

consequently, the negative root of the proposed equation is $-2\cdot35833$.

[12.] This equation, in a complete form, is

$$x^4 + 0x^3 + x^2 - 8x - 15 = 0;$$

and when the alternate signs are changed, it is

$$x^4 - 0x^3 + x^2 + 8x - 15 = 0;$$

the first member of which furnishes the factors,

$$x \pm \sqrt{(-x^4 - 8x + 15)};$$

and as the expression under the radical is obviously negative for $x=2$, and for all greater values, we commence with 1, as the first figure of the root, and develop as follows:

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 1 | 8 | 15 (1·302776 |
| | 1 | 1 | 2 | 10 |
| | 1 | 2 | 10 | 5 |
| | 1 | 2 | 4 | 4·9461 |
| | 2 | 4 | 14 | 539 |
| | ·1 | 3 | 2·487 | 38821 |
| | 3 | 7 | 16·487. | 15079 |
| | 1 | 1·29 | 2·901 | 13609 |
| | 4 | 8·29 | 19·388 | 1470 |
| | ·3 | 1·38 | 223 | 1361 |
| | 4·3 | 9·67 | 19·4103 | 109 |
| | ·3 | 1·47 | 223 | 117 |
| | 4·6 | 11·14 | 19·433 | |
| | ·3 | 104 | 8 | |
| | 4·9 | 11·1504 | 19·441 | |
| | ·3 | 1 | | |
| | 5·2 | 11·16 | | |
| | 02 | | | |
| | 5·202 | | | |

consequently the negative root of the proposed equation is $-1·302776$.

[13.] The equation in its complete form is,

$$x^4-2x^3+0x^2+3x-20=0.$$

and changing the alternate signs, we have to find the positive root of

$$x^4+2x^3+0x^2-3x-20=0.$$

A glance at the first member shows that it becomes negative for $x=1$, and positive for $x=2$: therefore the positive root lies between 1 and 2; that is, the leading figure of the root is 1.* . If, however, we apply the method of factors as in the foregoing examples, we have

$$x^2\pm\sqrt{(-2x^3+3x+20)}$$
$$=x^2-4\pm\sqrt{(-2x^3-8x^2+3x+36)};$$

---

* We here say *the* root, instead of *a* root, because we know, from a general principle in the theory of equations, that an equation cannot

where the expression under the radical is obviously negative for $x = 2$, and for all greater values.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 0 | −3 | 20 (1·876837 . . . |
| | 1 | 3 | 3 | 0 |
| | 3 | 3 | 0 | 20 (1) |
| | 1 | 4 | 7 | 16·7616 |
| | 4 | 7 | 7 (1) | 3·2384 (2) |
| | 1 | 5 | 13·952 | 2·93511 |
| | 5 | 12 (1) | 20·952 | ·30329 (3) |
| | 1 | 5·44 | 18·816 | ·26599 |
| | 6 (1) | 17·44 | 39·768 (2) | 3730 (4) |
| | ·8 | 6·08 | 2·1622 | 3562 |
| | 6·8 | 23·52 | 41·9302 | 168 |
| | ·8 | 6·72 | 2·208 | 134 |
| | 7·6 | 30·24 (2) | 44·138 (3) | 34 |
| | ·8 | ·649 | ·194 | 31 |
| | 8·4 | 30·889 | 44·332 | 3 |
| | ·8 | ·654 | ·194 | |
| | 9·2 (2) | 31·543 | 44·53 (4) | |
| | 7 | ·659 | | |
| | 9·27 | 32·20 (3) | | |
| | 7 | 6 | | |
| | 9·34 | 32·26 | | |
| | 7 | | | |
| | 9·41 | | | |
| | 7 | | | |
| | 9·5 (3) | | | |

consequently the negative root of the proposed equation is −1·876837. The seventh decimal is probably a 6; but when the operation is extended so far as to leave only a single

---

have a greater number of *positive* roots than there are changes of sign among the signs of its terms.

The signs in $x^4+2x^3\pm0x^2-3x-20=0$ change only *once*, namely, from + to −, and that whether we consider the zero term to have the one sign or the other; but for the proof of this useful principle, the learner may consult the treatise on "The Theory and Solution of Equations," by the Author of this Work, or "The Analysis and Solution of Cubic and Biquadratic Equations," which forms an introduction to the more extensive treatise just referred to.

figure for remainder in the last column, we cannot pronounce with confidence on the exact value of another decimal; for the final remainder, on account of the contractions, may err by a unit, and in extreme cases by two units: we may be certain, however, that the decimals found above are true as far as they go.

[14.] Changing the alternate signs, the equation becomes

$$x^4 - x^3 + x^2 - x - 1 = 0;$$

and for the factors of the first member, we have

$$x \pm \sqrt{(-x^4 + x^3 + x + 1)};$$

and as the expression under the radical is negative for $x = 2$, and for all greater values, we commence by assuming 1 for the first figure of the root.

| | | | | |
|---|---|---|---|---|
| 1 | −1 | 1 | −1 | 1(1·290648 |
| | 1 | 0 | 1 | 0 |
| | 0 | 1 | 0 | 1 (1) |
| | 1 | 1 | 2 | ·5856 |
| | 1 | 2 | 2 (1) | ·4144 (2) |
| | 1 | 2 | ·928 | ·41104 |
| | 2 | 4 (1) | 2·928 | 336 (3) |
| | 1 | ·64 | 1·064 | 311 |
| | 3 (1) | 4·64 | 3·992 (2) | 25 |
| | ·2 | ·68 | ·5751 | 21 |
| | 3·2 | 5·32 | 4·5671 | 4 |
| | ·2 | ·72 | ·6073 | 4 |
| | 3·4 | 6·04 (2) | 5·174 (3) | |
| | ·2 | ·350 | 4 | |
| | 3·6 | 6·390 | 5·178 | |
| | ·2 | ·358 | | |
| | 3·8 (2) | 6·748 | | |
| | 9 | ·366 | | |
| | 3·89 | 7·11 (3) | | |
| | 9 | | | |
| | 3·98 | | | |
| | 9 | | | |
| | 4·07 | | | |
| | 9 | | | |
| | 4·2 (3) | | | |

consequently the negative root of the proposed equation, as far as six places of decimals, is −1·290648. Whether, by postponing the contractions to a more advanced step of the work, and thus extending the root to a greater number of decimals, the seventh decimal would be sufficiently great to justify our changing the final 8 above into a 9, can be ascertained only by actual trial, that is, by extending the decimals in the several columns of work.

[15.] This equation, when the alternate signs are changed, is the same as the equation in Ex. 4, solved at page 162: the negative root is therefore −1·414214.

[16] This equation, too, when the alternate signs are changed, becomes that solved at page 215 of the Algebra; but we shall here show the operation at length, when applied directly to the development of the negative root, without converting it into a positive root.

| | | | | | *Negative Root.* |
|---|---|---|---|---|---|
| 1 | −1 | −29 | 27 | 6 | ( −5·372281 |
| | −5 | 30 | −5 | −110 | |
| | −6 | 1 | 22 | 116 (1) | |
| | −5 | 55 | −280 | 90·2151 | |
| | −11 | 56 | −258 (1) | 25·7849 (2) | |
| | −5 | 80 | −42·717 | 24·9457 | |
| | −16 | 136 (1) | −300·717 | ·8392 (3) | |
| | −5 | 6·39 | −44·661 | ·7356 | |
| | −21 (1) | 142·39 | −345·378 (2) | ·1036 (4) | |
| | −·3 | 6·48 | −10·989 | 736 | |
| | −21·3 | 148·87 | −356·367 | 300 | |
| | −·3 | 6·57 | −11·10 | 294 | |
| | −21·6 | 155·44 (2) | −367·47 (3) | 6 | |
| | −·3 | 1·55 | −·32 | 4 | |
| | −21·9 | 156·99 | −367·79 | 2 | |
| | −·3 | 1·5 | −·3 | | |
| | −22·2 (2) | 158·5 | 368·1 (4) | | |
| | | 1 | | | |
| | | 16·0 (3) | | | |

From this operation the learner will perceive, that by changing the alternate signs in an equation, and by also changing the sign of the root as given by the unmodified equation, the only difference in the operation is, that the numbers in alternate columns of work all have changed signs, the *values* remaining undisturbed, which shows that by changing the alternate signs in the original equation, the only effect is to change the signs of the roots of that equation; and a little reflection is sufficient to convince us that, in every equation, such must necessarily be the case. But, for a more formal proof of the proposition, the inquiring student is directed to the sources of information already referred to (page 173). We have, in truth, in the foregoing solutions, and in the remarks and explanations which have accompanied them, sufficiently unfolded to the learner what appears to us to be the shortest and most effective method of developing, figure by figure, any real root of a numerical equation. He must have perceived that, in general, the most troublesome step in the process is the leading step—that by which the *first* figure of the root is discovered. Without the aid afforded to him by the method of decomposing a polynomial into factors, so generally employed in the preceding solutions, he would have found the determination of the initial-root figure to have involved him, in some cases, in a good deal of weary guess-work; and although, even with the aid here offered, trial numbers, which on being put to the test are found to fail, are not always to be dispensed with, yet there is great advantage in knowing, with absolute certainty, that, beyond a certain clearly indicated limit, no such trial-numbers need ever be selected; and it is this advantage that the method offers, by pointing out, in the most unambiguous manner, a definite and comparatively narrow region, within the boundaries of which the number sought must necessarily be situated. Of late years a great deal has been attempted in the way of devising practicable and certain rules, for discovering the situations of the real roots of numerical equations. A comprehensive view of what has hitherto been accomplished, in this important field of algebraical research, will be found in "The General Theory and Solution of Algebraical Equations" before mentioned.

## ARITHMETICAL PROGRESSION.

### PAGE 226.

[1.] IN this series, the particular values of the general symbols in article 108 are as follows; namely,

$$a=1,\ \ d=3,\ \ n=12;\ \ \therefore l=a+(n-1)d=34;$$
$$\therefore S=\tfrac{1}{2}n(a+l)=6(1+34)=210.$$

[2.] In this series $d$ is negative, and the values of the symbols are,

$$a=2,\ \ d=-\tfrac{1}{2},\ \ n=14;\ \ \therefore l=a+(n-1)d=-4\tfrac{1}{2};$$
$$\therefore S=\tfrac{1}{2}n(a+l)=7(2-4\tfrac{1}{2})=-17\tfrac{1}{2}.$$

[3.] The general expression for the sum is

$$S=\tfrac{1}{2}n\{2a+(n-1)d\};$$

which, upon substituting the particular values given in the example, is $\quad 14=\tfrac{7}{2}\{22+6d\},$

an equation in which $d$ is the unknown quantity;

$$\therefore 14=7(11+3d);\ \ \therefore 21d=14-77=-63;$$
$$\therefore d=-\tfrac{63}{21}=-3.$$

[4.] Taking the general expression for the sum, namely,

$$S=\tfrac{1}{2}n\{2a+(n-1)d\},$$

and substituting the given values for the symbols, we have

$$150=12\{1+23d\},\ \text{or}\ 150=12+276d;$$
$$\therefore d=\tfrac{138}{276}=\tfrac{1}{2}.$$

[5.] Here the first term of the series is $a=\frac{1}{3}$, the last, $l=\frac{5}{6}$, the number of terms, $n=7$; and it is required to find the common difference $d$. The formula for $l$ is,

$$l=a+(n-1)d;\ \ \therefore \tfrac{5}{6}=\tfrac{1}{3}+6d;$$
$$\therefore d=\tfrac{1}{6}(\tfrac{5}{6}-\tfrac{1}{3})=\tfrac{1}{6}\cdot\tfrac{1}{2}=\tfrac{1}{12};$$

consequently the five intermediate terms are,

$$\tfrac{5}{12},\ \tfrac{1}{2},\ \tfrac{7}{12},\ \tfrac{2}{3},\ \tfrac{3}{4}.$$

[6.]

Here the first term $a = \frac{1}{2}$, the difference $d = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$, and the sum $S = 0$, are given to find the number of terms $n$.

The formula is, $S = \frac{1}{2}n\{2a + (n-1)d\}$;

$$\therefore 0 = \tfrac{1}{2}n\{1 - \tfrac{1}{6}(n-1)\};$$

$$\text{that is,}\quad 0 = \frac{n}{2} - \frac{n^2}{12} + \frac{n}{12};$$

therefore, multiplying by 12, dividing by $n$, and transposing, $n = 6 + 1 = 7$; consequently the sum of seven terms of the proposed series will be 0.

[7.] In the first case, $a = 1$, $l = -1$, and $n = 8$; so that $l = a + (n-1)d$ becomes $-1 = 1 + 7d$; $\therefore d = -\frac{2}{7}$; consequently the six means are, $\frac{5}{7}$, $\frac{3}{7}$, $\frac{1}{7}$, $-\frac{1}{7}$, $-\frac{3}{7}$, $-\frac{5}{7}$.

In the second case, $a = \frac{1}{2}$, $l = -\frac{1}{2}$, and $n = 7$; so that $l = a + (n-1)d$ becomes $-\frac{1}{2} = \frac{1}{2} + 6d$; $\therefore d = -\frac{1}{6}$; consequently the five means are $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$.

[8.] Here we have given the number of terms, $n = 21$, the common difference, $d = \frac{1}{5}$, and the sum, $S = -28$, to find $a$, the first term. The formula for this purpose is,

$$S = \tfrac{1}{2}n\{2a + (n-1)d\};$$

$$\therefore -28 = \tfrac{21}{2}\{2a + \tfrac{20}{5}\};$$

$$\text{or,}\ -28 = 21a + 42;\ \therefore a = -\tfrac{70}{21} = -3\tfrac{1}{3};$$

therefore, the first term of the series is $-3\frac{1}{3}$.

[9.] Here are given the first term, $a = \frac{1}{2}$, the difference, $d = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$, and the sum, $S = -1\frac{1}{2}$, to find $l$, the last term. The formula for finding the number of terms $n$ from these given quantities is,

$$S = \tfrac{1}{2}n\{2a + (n-1)d\};$$

$$\therefore -\tfrac{3}{2} = \tfrac{1}{2}n\{1 - \tfrac{1}{6}(n-1)\};$$

$$\text{or, multiplying by 2,}\ -3 = n - \frac{n^2}{6} + \frac{n}{6};$$

multiplying by 6 and transposing,

$$n^2 - 7n = 18;\ \therefore 2n - 7 = \sqrt{(72 + 49)} = \sqrt{121} = \pm 11;$$

$$\therefore\ n=\frac{7\pm 11}{2}=9,\ \text{or}\ -2.$$

Taking the first value of $n$, we have

$$l=a+(n-1)d=\tfrac{1}{2}-\tfrac{8}{6}=-\tfrac{5}{6};$$

hence the last term of the series will be the ninth, the value of this term will be $-\frac{5}{6}$, and the sum of the whole $-1\frac{1}{2}$.

Although two values of $n$ satisfy the equation, it is plain, from the restrictions of the question, that only one of them, the positive value, can be applicable.

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## Geometrical Progression.—Page 229.

[1.] By the general formulæ, $l=ar^{n-1}$, $S=\dfrac{rl-a}{r-1}$.

In the example, $a=1$, $r=2^2=4$, and $n=6$;

$$\therefore\ l=4^5=1024;\ \therefore\ S=\frac{4096-1}{3}=1365.$$

[2.] Here $a=1$, $r=-4$, and $n=5$,

$$\therefore\ l=(-4)^4=256;\ \therefore\ S=\frac{-1024-1}{-5}=205.$$

[3.] Here $a=5$, $r=-3$, and $n=7$;

$$\therefore\ l=5(-3)^6=5.27^2=3645;$$

$$\therefore\ S=\frac{-10935-5}{-4}=2735.$$

[4.] Here $a=\frac{1}{3}$, $r=\frac{1}{2}$, and $n=6$;

$$\therefore\ l=\tfrac{1}{3}(\tfrac{1}{2})^5=\tfrac{1}{3}\cdot\tfrac{1}{32}=\tfrac{1}{96};$$

$$\therefore\ S=\frac{\frac{1}{2}\cdot\frac{1}{96}-\frac{1}{3}}{-\frac{1}{2}}=-\tfrac{1}{96}+\tfrac{2}{3}=-\tfrac{1}{96}+\tfrac{64}{96}=\tfrac{63}{96}=\tfrac{21}{32}.$$

[5.] Here $a=\frac{1}{3}$, $r=-\frac{1}{2}$, and $n=6$;

$$\therefore\ l=\tfrac{1}{3}(-\tfrac{1}{2})^5=-\tfrac{1}{3}\cdot\tfrac{1}{32}=-\tfrac{1}{96};$$

$$\therefore\ S=\frac{\frac{1}{2}\cdot\frac{1}{96}-\frac{1}{3}}{-\frac{3}{2}}=\frac{\frac{1}{96}-\frac{2}{3}}{-3}=\tfrac{21}{96}=\tfrac{7}{32}.$$

[6.] Here $a = 1$, $r = \frac{1}{4}$; therefore by the formula,

$$\Sigma = \frac{a}{1-r} = \frac{1}{\frac{3}{4}} = \tfrac{4}{3} = 1\tfrac{1}{3}.$$

[7.] Here $a = \frac{1}{2}$, $r = -\frac{1}{2}$; therefore by the formula,

$$\Sigma = \frac{a}{1-r} = \frac{\frac{1}{2}}{\frac{3}{2}} = \tfrac{1}{2} \times \tfrac{2}{3} = \tfrac{1}{3}.$$

[8.] Here $a = 1$, $r = \frac{1}{x}$; therefore

$$\Sigma = \frac{a}{1-r} = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1}.$$

[9.] Here $a = \frac{3}{4}$, $r = -\frac{3}{4}$; therefore,

$$\Sigma = \frac{a}{1-r} = \frac{\frac{3}{4}}{\frac{7}{4}} - \tfrac{3}{4} \times \tfrac{4}{7} = \tfrac{3}{7}.$$

[10.] $\sqrt{(\frac{1}{2} \times \frac{1}{8})} = \sqrt{\frac{1}{16}} = \frac{1}{4}$, the mean between $\frac{1}{2}$ and $\frac{1}{8}$; $\sqrt{(\frac{3}{4} \times \frac{25}{27})} = \sqrt{\frac{25}{4 \times 9}} = \frac{5}{6}$, the mean between $\frac{3}{4}$ and $\frac{25}{27}$.

[11.] The last term of a geometrical series being $l = ar^{n-1}$, the fourth term of the series in the present example is $108 = 4r^3$; $\therefore 27 = r^3$; $\therefore 3 = r$; that is; the common ratio is 3; consequently the two wanting terms are 12 and 36.

[12.] The formula $l = ar^{n-1}$, in the first case of the present example, is $\frac{1}{16} = r^4$; $\therefore \frac{1}{2} = r$, the common ratio; consequently the three means are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$.

In the second case the formula becomes

$$\tfrac{2}{9} = \tfrac{1}{2}; \therefore r^4; \therefore r^4 = \tfrac{4}{9};\ r^2 = \tfrac{2}{3}; \therefore r = \sqrt{\tfrac{2}{3}}, \text{ or } \tfrac{1}{3}\sqrt{6};$$

consequently the three means are

$$\tfrac{1}{2}\sqrt{\tfrac{2}{3}},\ \tfrac{1}{3},\ \tfrac{1}{3}\sqrt{\tfrac{2}{3}}; \text{ or } \tfrac{1}{6}\sqrt{6},\ \tfrac{1}{3},\ \tfrac{1}{9}\sqrt{6}.$$

## SIMULTANEOUS QUADRATIC EQUATIONS.

### PAGE 232.

$$[1.] \quad \left.\begin{aligned} x^2 + xy &= 77 \\ xy - y^2 &= 12 \end{aligned}\right\}$$

Putting $zx$ for $y$, these equations become

$$x^2 + zx^2 = 77; \ \therefore x^2 = \frac{77}{1+z} \ \ldots (A);$$

$$zx^2 - z^2x^2 = 12; \ \therefore x^2 = \frac{12}{z - z^2};$$

$$\therefore \frac{12}{z-z^2} = \frac{77}{1+z}; \ \therefore 12 + 12z = 77z - 77z^2;$$

$$\text{transposing,} \quad 77z^2 - 65z = -12;$$

$\therefore$ Rule II. page 178, Algebra,

$$154z - 65 = \sqrt{(-3696 + 4225)} = \sqrt{529} = \pm 23;$$

$$\therefore z = \frac{65 \pm 23}{154} = \tfrac{4}{7}, \text{ or } \tfrac{3}{11}.$$

Substituting the first of these values in $(A)$, we have

$$x^2 = \frac{77}{1 + \frac{4}{7}} = \frac{77}{\frac{11}{7}} = 49; \ \therefore x = \pm 7;$$

$$\therefore y = zx = \pm 7 \times \tfrac{4}{7} = \pm 4;$$

and if $\frac{3}{11}$ be substituted for $z$ in the same equation $(A)$, another pair of values may be obtained.

$$[2.] \quad \left.\begin{aligned} x^2 + xy + y^2 &= 52 \\ xy - x^2 &= \ \ 8 \end{aligned}\right\}$$

Putting $zx$ for $y$, the equations become

$$x^2 + zx^2 + z^2x^2 = 52; \ \therefore x^2 = \frac{52}{1 + z + z^2};$$

$$x^2z - x^2 = 8; \ \ \therefore x^2 = \frac{8}{z-1} \ \ldots (A);$$

$$\therefore \frac{8}{z-1} = \frac{52}{1+z+z^2}; \ \therefore 8 + 8z + 8z^2 = 52z - 52;$$

transposing, $8z^2 - 44z = -60$; $\therefore 2z^2 - 11z = -15$;

$\therefore$ Rule II. page 178, $4z - 11 = \sqrt{(-120 + 121)} = \pm 1$;

$$\therefore z = \frac{11 \pm 1}{4} = 3, \text{ or } \tfrac{5}{2};$$

$$\therefore (A),\ x^2 = \frac{8}{3 - 1} = 4;$$

$$\therefore x = \pm 2; \ \therefore y = zx = \pm 3 \times 2 = \pm 6;$$

and if $\frac{5}{2}$ be put for $z$ in $(A)$, another pair of values may be obtained.

$$[3.] \quad \left.\begin{aligned} 2x^2 - \ \ xy &= 6 \\ 2y^2 + 3xy &= 8 \end{aligned}\right\}$$

Putting $zx$ for $y$, the equations become

$$2x^2 - zx^2 = 6; \ \therefore x^2 = \frac{6}{2 - z} \ldots (A);$$

$$2z^2x^2 + 3zx^2 = 8; \ \therefore x^2 = \frac{8}{2z^2 + 3z};$$

$$\therefore \frac{3}{2 - z} = \frac{4}{2z^2 + 3z}; \ \therefore 6z^2 + 9z = 8 - 4z;$$

transposing, $6z^2 + 13z = 8$;

$\therefore$ Rule II. page 178,

$$12z + 13 = \sqrt{(192 + 169)} = \sqrt{361} = \pm 19;$$

$$\therefore z = \frac{-13 \pm 19}{12} = \tfrac{1}{2}, \text{ or } -\tfrac{8}{3};$$

$$\therefore (A),\ x^2 = \frac{6}{2 - \frac{1}{2}} = 4;$$

$$\therefore x = \pm 2; \ \therefore y = zx = \pm\tfrac{1}{2} \times 2 = \pm 1.$$

If the other value of $z$ be substituted in $(A)$, we shall have

$$x^2 = \frac{6}{2 + \frac{8}{3}} = 6 \times \tfrac{3}{14} = \tfrac{9}{7};$$

$$\therefore x = \pm \frac{3}{\sqrt{7}} = \pm\tfrac{3}{7}\sqrt{7}; \ \therefore y = zx = -\tfrac{8}{3} \times \pm\tfrac{3}{7}\sqrt{7} = \mp\tfrac{8}{7}\sqrt{7}.$$

[4.] $$\left.\begin{aligned} x+y &= 5 \\ \frac{x}{y}+\frac{y}{x} &= \frac{17}{4} \end{aligned}\right\}$$

These equations are solved in the Algebra, page 233; they may be treated in a different way, as follows:

Put $z$ for $\frac{x}{y}$; then the second equation becomes,

$$z+\frac{1}{z}=\frac{17}{4};\ \therefore 4z^2+4=17z;\ \therefore 4z^2-17z=-4;$$

$\therefore$ Rule II. page 178,

$$8z-17=\surd(-64+289)=\surd 225=\pm 15;$$

$$\therefore z=\frac{17\pm 15}{8}=4,\text{ or }\tfrac{1}{4};$$

$$\text{that is, }\frac{x}{y}=4,\text{ or }\tfrac{1}{4};\ \therefore x=4y,\text{ or }\frac{y}{4}\ldots(A);$$

consequently, by substitution, the first equation becomes either $4y+y=5$, or $\frac{y}{4}+y=5$.

From the former of these, $y=1$; from the latter, $y=4$;

$$\therefore (A),\ x=4,\text{ or }1.$$

[5.] Let $x$, $y$, represent the numbers; then by the question, supposing $x$ to be the greater number,

$$x^2+y^2=89,\ x(x+y)=104.$$

Put $y=zx$; then these equations become,

$$\begin{aligned} x^2+z^2x^2 &= 89 \\ x^2+zx^2 &= 104; \end{aligned}$$

$$\therefore x^2=\frac{89}{1+z^2},\text{ and }x^2=\frac{104}{1+z}\ldots\therefore (A);$$

$$\therefore \frac{104}{1+z}=\frac{89}{1+z^2};\ \therefore 104+104z^2=89+89z;$$

transposing, $104z^2-89z=-15$; $\therefore$ Rule II. page 178,

$$208z-89=\surd(-6240+7921)=\surd 1681=\pm 41;$$

$$\therefore z=\frac{89\pm 41}{208}=\tfrac{5}{8},\text{ or }\tfrac{3}{13};$$

$$\therefore (A),\ x^2 = \frac{104}{1+\frac{5}{8}} = 104 \times \tfrac{8}{13} = 64; \quad \therefore x = \pm 8;$$

$$\text{or, } x^2 = \frac{104}{1+\frac{3}{13}} = 104 \times \tfrac{13}{16} = \frac{13 \times 13}{2};\ \therefore x = \frac{13\sqrt{2}}{2}.$$

The values of $y$, corresponding to the first pair of values of $x$, are $y = zx = \pm\frac{5}{8} \times 8 = \pm 5$; and those corresponding to the second pair are $y = zx = \frac{3}{13} \times \frac{13\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$.

[6.] Let the three numbers be represented by $x-y$, $x$, $x+y$, the first being the least, and consequently the third the greatest, then by the question,

$$(x-y)^2 + x(x+y) = 28$$
$$(x+y)^2 + x(x-y) = 44;$$

adding and subtracting, we have

$$4x^2 + 2y^2 = 72; \quad \therefore 2x^2 + y^2 = 36 \ldots (A)$$

$$2xy = 16; \ \therefore xy = 8; \ \therefore y = \frac{8}{x} \ \ldots (B);$$

hence, by substitution in $(A)$, we get

$$2x^2 + \frac{64}{x^2} = 36; \ \therefore 2x^4 + 64 = 36x^2; \ \therefore x^4 + 32 = 18x^2;$$

that is, putting $z$ for $x^2$, and transposing $z^2 - 18z = -32$; completing the square, $z^2 - 18z + 81 = 49$; $\therefore z - 9 = \pm 7$;

$$\therefore z = 16, \text{ or } 2.$$

Taking the first of these values, we have $x = \pm 4$; consequently $y = \frac{8}{x} = \pm 2$: hence the three numbers are $\pm 2$, $\pm 4$, $\pm 6$; and these are the only *rational* members which fulfil the conditions of the question.

*Otherwise.* As $(A)$, $(B)$, are homogeneous equations, we may apply the general method of the present article, and put $zx$ for $y$; we thus have

$$2x^2 + z^2x^2 = 36, \ \therefore x^2 = \frac{36}{2+z^2}$$

$$zx^2 = 8, \quad \therefore x^2 = \frac{8}{z} \ldots (A);$$

$$\therefore \frac{8}{z} = \frac{36}{2+z^2}; \quad \therefore 4 + 2z^2 = 9z;$$

transposing,

$$2z^2 - 9z = -4; \quad \therefore 4z - 9 = \sqrt{(-32+81)} = \sqrt{49} = \pm 7;$$

$$\therefore z = \frac{9 \pm 7}{4} = 4, \text{ or } \tfrac{1}{2};$$

$$\therefore (A),\ x^2 = \frac{8}{4} = 2, \text{ or } \frac{8}{\frac{1}{2}} = 16;$$

$$\therefore x = \sqrt{2}, \text{ or } \pm 4;$$

therefore, taking account only of rational values, $y = zx = \pm 4 \times \frac{1}{2} = \pm 2$; consequently the three numbers are $\pm 2$, $\pm 4$, $\pm 6$.

[7.] Let $x, y$ be the two numbers, then, by the question, $xy = 12$, and $x^2 + y^2 : x^2 - y^2 :: 25 : 7$;

$$\therefore 7x^2 + 7y^2 = 25x^2 - 25y^2 \text{ (Algebra, p. 222, art. 105)};$$

$$\therefore 32y^2 = 18x^2, \text{ or } 16y^2 = 9x^2;$$

$$\therefore 4y = 3x; \qquad \therefore y = \frac{3x}{4};$$

substituting this in the equation $xy = 12$, we have

$$\frac{3x^2}{4} = 12; \quad \therefore x^2 = 16; \quad \therefore x = \pm 4;$$

$$\therefore y = \frac{3x}{4} = \pm 3;$$

consequently the numbers are $\pm 3$ and $\pm 4$.

*Otherwise.* Putting $zx$ for $y$ in $4y = 3x$, and in $xy = 12$, we have

$$4zx = 3x; \quad \therefore z = \tfrac{3}{4}$$

$$zx^2 = 12; \quad \therefore z = \frac{12}{x^2};$$

$$\therefore \tfrac{1}{4} = \frac{4}{x^2}; \quad \therefore x^2 = 16; \quad \therefore x = \pm 4;$$

$$\therefore y = zx = \pm 4 \times \tfrac{3}{4} = \pm 3.$$

[8.] Let the four numbers be represented by

$$x,\ x+y,\ x+2y,\ x+3y;$$

then, by the question, $x^2+3xy=45\ldots(A)$

$$x^2+3xy+2y^2=77\ldots(B);$$

subtracting the first from the second, we have

$$2y^2=32;\ \therefore y^2=16;\ \therefore y=\pm 4;$$

consequently the first equation is

$$x^2\pm 12x=45;$$

∴ Rule I. completing the square,

$$x^2\pm 12x+36=81;$$

extracting the root, $x\pm 6=\pm 9$; $\therefore x=\pm 9\mp 6$;

so that the only *positive* value of $x$ is 3; and since that of $y$ is found above to be 4, the arithmetical progression required is 3, 7, 11, 15.

The homogeneous equations ($A$), ($B$), may of course be solved by putting $zx$ for $y$; but that here given is the more simple solution.

[9.] Let $x, y$ represent the numbers, then, by the question,

$$x^2+y^2=13001\ldots(A)$$
$$x^2-y^2=1449\ldots(B);$$

adding, $2x^2=14450$; $\therefore x^2=7225$; $\therefore x=\pm 85$;

$\therefore (A)$, $y^2=13001-7225=5776$; $\therefore y=\pm 76$;

consequently the two numbers are $\pm 85$, $\pm 76$.

This is, of course, an easier solution than would be obtained by employing the general method of the present article.

[10.] Let the three numbers be represented by

$$x-y,\ x,\ x+y;$$

then, by the question,

$$(x-y)^2+x^2+(x+y)^2=93\ldots(A)$$
$$3(x-y)+4x+5(x+y)=66\ldots(B);$$

from ($B$), $12x+2y=66$; $\therefore y=33-6x$;

substituting in ($A$),

$$(7x-33)^2+x^2+(33-5x)^2=93;$$

$$\text{that is,}\quad 75x^2 - 792x + 2(33)^2 = 93;$$

$$\therefore\ 25x^2 - 264x + 2 \times 11 \times 33 = 31;$$

$$\therefore\ 25x^2 - 264x = -695;$$

$\therefore$ Rule II., p. 178, $50x - 264 = \sqrt{(-69500 + 69696)}$

$$= \sqrt{196} = \pm 14;$$

$$\therefore\ x = \frac{264 \pm 14}{50} = 5 \text{ or } \tfrac{139}{25};$$

taking the first of these values, we have

$$y = 33 - 6x = 3;$$

$$\therefore\ x - y = 2, \text{ and } x + y = 8;$$

therefore the numbers are 2, 5, 8.

[11.] Let $x$, $y$ represent the numbers; then, by the question,

$$x + y = xy = x^2 + y^2;$$

$$\begin{array}{lrcl} & \therefore\ 2x + 2y & = & 2xy \\ \text{adding,} & \overline{3(x+y)} & = & \overline{(x+y)^2}; \end{array}$$

$$\therefore\ 3 = x + y = xy \ldots (A);$$

$$\begin{array}{lrcl} & \therefore\ 9 & = & (x+y)^2 \\ & 12 & = & 4xy \\ \text{subtracting,} & \overline{-3} & = & \overline{(x-y)^2}; \end{array}$$

$$\therefore\ \sqrt{-3} = x - y \ldots (B);$$

adding and subtracting ($A$), ($B$),

$$3 + \sqrt{-3} = 2x,\ 3 - \sqrt{-3} = 2y;$$

$$\therefore\ x = \tfrac{1}{2}(3 + \sqrt{-3}),\ y = \tfrac{1}{2}(3 - \sqrt{-3}).$$

These are two imaginary quantities, showing that no real numbers satisfying the conditions of the question exist.

[12.] Let the digits be represented by $x$, $y$, $z$; then the conditions of the question are as follows:

$$100x + 10y + z : x + y + z :: 124 : 7 \ldots\ldots (1)$$

$$100x + 10y + z + 594 = 100z + 10y + x \ldots (2)$$

$$y^2 = xz \ldots\ldots\ldots\ldots (3)$$

The proportion (1) may be replaced by the following, namely,

$$99x+9y : x+y+z :: 117 : 7;^*$$

$$\text{or, } 11x+y : x+y+z :: 13 : 7;$$

$$\therefore 77x+7y=13x+13y+13z \ldots (4);$$

also from (2), $99z-99x=594$; $\therefore z-x=6$; $\therefore z=6+x$; consequently, (4) and (3), the three equations of condition, are

$$64x-6y=13z \ldots\ldots (A)$$

$$z=6+x \ldots (B)$$

$$y^2=xz \ldots\ldots (C);$$

substituting (*B*) in both (*A*) and (*C*), we have

$$64x-6y=78+13x; \therefore 51x-6y=78; \therefore 17x-2y=26,$$

$$y^2=6x+x^2; \therefore 4y^2=24x+4x^2;$$

but $4y^2=(17x-26)^2$, by the former equation,

$$\therefore (17x-26)^2=4x^2+24x;$$

$$\therefore 285x^2-908x=-676;$$

$\therefore$ Rule II., page 178, $570x-908=\sqrt{(-770604+824464)}$
$=\sqrt{53824}=\pm 232;$

$$\therefore x=\frac{908\pm 232}{570}=2;$$

$$\therefore z=6+x=8; \therefore y=\sqrt{xz}=4;$$

consequently the number is 248.

*Otherwise.* Let $x$ be the first of the three digits, and $y$ the common ratio, then the digits will be expressed by $x$, $xy$, $xy^2$; and, by the question,

$$100x+10xy+xy^2 : x+xy+xy^2 :: 124 : 7;$$

$$\therefore 100+10y+y^2 : 1+y+y^2 :: 124 : 7;$$

$$\therefore 99+9y : 1+y+y^2 :: 117 : 7;^*$$

$$\therefore 11+y : 1+y+y^2 :: 13 : 7;$$

---

* Any proportion $a:b::c:d$, is equivalent to an equation $\frac{a}{b}=\frac{c}{d}$; and by subtracting 1 from each member of this, we have $\frac{a}{b}-1=\frac{c}{d}-1$; that is, $\frac{a-b}{b}=\frac{c-d}{d}$; $\therefore a-b:b::c-d:d$, as in the text. The principle is also established at page 113 of the Algebra.

$$\therefore 13y^2 + 13y + 13 = 7y + 77;$$
$$\therefore 13y^2 + 6y = 64;$$

$\therefore$ Rule II., page 178, $26y + 6 = \sqrt{(3328 + 36)} = \sqrt{3364}$
$= \pm 58;$

$$\therefore y = \frac{-6 \pm 58}{26} = 2, \text{ or } -\tfrac{32}{13};$$

also, by the question,

$$100x + 10xy + xy^2 + 594 = 100xy^2 + 10xy + x;$$

transposing, $\quad 99x + 594 = 99xy^2;$

$$\therefore x + 6 = xy^2 = 4x;$$
$$\therefore 3x = 6; \ \therefore x = 2;$$
$$\therefore xy = 4, \quad xy^2 = 8;$$

consequently the number is 248.

The solutions to the examples in the Algebra terminate here. Those who wish to prosecute the subject further, especially in reference to the theory and solution of Equations, will find the fullest information, and the most recent improvements, in the author's "Analysis and Solution of Cubic and Biquadratic Equations," and in his larger work on "Equations of the Higher Orders."

The two supplementary notes which follow, may also be worthy of the student's attention. The first of these exhibits, in a convenient and somewhat novel form, the general treatment of the subject of Elimination in Simultaneous Equations of the first degree; and the second furnishes some additional particulars that may be found of use in the decomposition of Biquadratic Equations. The principles upon which this decomposition of Equations is founded, were first published by the author in Part I. of the "General Principles of Analysis," referred to at page 219 of the Algebra; and it is to this work that reference must be made for the application of the method to equations of the more advanced degrees.

# NOTE (A.) PAGE 108.

## ON ELIMINATION IN SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

AT page 137 of the Algebra, sufficient explanation is given of the method of elimination as applied to three simultaneous simple equations, when the coefficients in those equations are numbers.

By following the directions there given, it is obvious that general formulæ might be obtained for the three unknown quantities, when the coefficients are letters instead of numbers; just as general formulæ were determined at page 134, for the unknowns in two simultaneous equations with literal coefficients. Like formulæ, for the unknowns in three equations, we omitted to introduce for two reasons: first, because the solution of an example, in which the coefficients are all given numbers, is as readily effected by applying the general precepts to the particular case, as by substituting the particular values in the literal formulæ; while the actual operation, applied to each individual example, is a more instructive exercise for a learner, than the mere translation of letters into numbers. A second reason for omitting the general formulæ is, that they are difficult to carry in the memory, while the operations, the results of which they embody, are very easily retained; for the successive steps in a great measure suggest themselves.

But, notwithstanding all this, the general formulæ are sometimes wanted, though not in common algebra; for there are many inquiries in which the relations of *coefficients* are more the objects of search, than the *values* of the quantities with which these coefficients are connected: we therefore think it right to record the general formulæ for the unknowns in three simultaneous equations here; and, moreover, to present the learner with an easily remembered process for recovering them at any time, without a formal reference to a book. *Operations* in which the eye and the fingers take part, as well as the mind, are usually more distinctly impressed on the memory than mere formulæ. A person is less likely to forget the process for solving a quadratic equation, for instance, than the general form for the result of that process.

If we express three simultaneous simple equations as follows, namely,

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

we shall find, by proceeding according to the precepts at page 137, that the values of the three unknown quantities will be,

$$x = \frac{(b_1c_3 - c_1b_3)d_2 + (b_2c_1 - c_2b_1)d_3 + (b_3c_2 - c_3b_2)d_1}{(b_1c_3 - c_1b_3)a_2 + (b_2c_1 - c_2b_1)a_3 + (b_3c_2 - c_3b_2)a_1}$$

$$y = \frac{(a_1c_3 - c_1a_3)d_2 + (a_2c_1 - c_2a_1)d_3 + (a_3c_2 - c_3a_2)d_1}{(a_1c_3 - c_1a_3)b_2 + (a_2c_1 - c_2a_1)b_3 + (a_3c_2 - c_3a_2)b_1}$$

$$z = \frac{(a_1b_3 - b_1a_3)d_2 + (a_2b_1 - b_2a_1)d_3 + (a_3b_2 - b_3a_2)d_1}{(a_1b_3 - b_1a_3)c_2 + (a_2b_1 - b_2a_1)c_3 + (a_3b_2 - b_3a_2)c_1}$$

These formulæ, obtained as directed, are not reached without a good deal of algebraical work; but we may arrive at them, with comparatively little trouble, by an easily remembered process, as follows:—

1. Having fixed upon the unknown to be determined, draw a line under those terms of the last equation which involve the other two unknowns.

2. Multiply *crosswise* the coefficients, above this line, of the first and second equations, separating the resulting products by the minus sign. Do the same with the second and third equations, and then with the first and third; but, in this last operation, regard the first equation to be brought down under the third, before the cross multiplication is performed, in order that each binomial product may have its signs regulated by a uniform principle—namely, that when the multiplier is on the left, and the multiplicand on the right, the resulting product is *plus*; and when the multiplier is on the right, and the multiplicand on the left, the product is *minus*.

This is all that need be made mere matter of memory. The three binomial expressions thus obtained will furnish the numerator of the sought value, when each binomial has connected with it, as a factor, the final term ($d_1$, $d_2$, or $d_3$) of that equation which is not used in forming the binomial. The denominator will consist of the same binomials; but the factor belonging to each will here be that coefficient, in the same unused equation which is prefixed to the unknown whose value is to be determined.

The binomials being obtained by cross multiplication, as just directed, the correct factors of them, whether in numerator or denominator, can require no effort of memory; for a mere inspection will show which of the three factors, in either case, has claim to be introduced, to the exclusion of the others.

Guided by these precepts, we shall now proceed to determine the value of $z$. The terms involving $x$ and $y$, the other unknowns, are the only terms written down, as the coefficients of these alone enter into the formation of the binomials.

$$\begin{array}{l}
a_1x+b_1y\\
a_2x+b_2y\\
\underline{a_3x+b_3y}\\
a_2b_1-b_2a_1\\
a_3b_2-b_3a_2\\
a_1b_3-b_1a_3
\end{array}$$

These are the binomials which enter alike into numerator and denominator in the formula for $z$. For the numerator, they are to be multiplied by $d_3$, $d_1$, $d_2$, respectively; and for the denominator, by $c_3$, $c_1$, $c_2$.

As observed above, it is plain, from inspection, that this is the proper order of the factors; for if any other order were for a moment imagined to be correct, mere symmetry of position would forbid the arrangement. We could not, for instance, in forming the numerator, connect the binomial $a_2b_1-b_1a_2$, with $d_1$ or $d_2$, seeing that each has *equal* claim to be selected, since *both* belong to the equations used in forming that binomial—a circumstance which precludes a preference for either; and, on like grounds, in forming the denominator, this same binomial cannot take for factor either $c_1$, or $c_2$. The value of $z$ is thus very rapidly found to be the expression exhibited above; the order in which the binomials are connected together being, of course, of no moment.

Proceeding in a similar manner for the determination of $y$, we have

$$\begin{array}{l}
a_1x+c_1z\\
a_2x+c_2z\\
\underline{a_3x+c_3z}\\
a_2c_1-c_2a_1\\
a_3c_2-c_3a_2\,.\\
a_1c_3-c_1a_3
\end{array}$$

These are the binomials which enter both numerator and denominator of the expression for $y$. For the numerator, they are to be multiplied by $d_3$, $d_1$, $d_2$, respectively; and for the denominator, by $b_3$, $b_1$, $b_2$.

Lastly, for the binomials which enter into the expression for $z$, we proceed in the same way:

$$\begin{array}{l} b_1y + c_1z \\ b_2y + c_2z \\ \underline{b_3y + c_3z} \\ b_2c_1 - c_2b_1 \\ b_3c_2 - c_3b_2 \\ b_1c_3 - c_1b_3 \end{array}$$

And these are to be multiplied by $d_3$, $d_1$, $d_2$, respectively, for the numerator; and by $a_3$, $a_1$, $a_2$, for the denominator.

Although, as before stated, the foregoing process is recommended chiefly on account of the facility with which it enables us to write the above general forms for $x$, $y$, and $z$, when the coefficients in the proposed equations are letters, it is, of course, equally applicable when the coefficients are numbers, and is indeed shorter than the methods in common use. In applying it in such particular examples, we may suppress any numerical factor which may happen to be common to the coefficients in either of the two vertical rows that supply the binomials; and these binomials themselves may also be freed from common factors. The following is an example taken at random—namely, to find $z$ from the following equations:

$$\begin{array}{l} 2x + 4y - 3z - 22 = 0 \\ 4x - 2y + 5z - 18 = 0 \\ 6x + 7y - \phantom{3}z - 63 = 0 \end{array}$$

Suppressing the factor 2, common to the first vertical row of coefficients, we proceed, for the determination $z$, thus,

$$\begin{array}{rcrl} 1 + \phantom{1}4 & & & \\ 2 - \phantom{1}2 & & & \\ \underline{3 + \phantom{1}7} & & & \\ 8 + 2 & = & 10 & \text{or } \phantom{-}2 \\ -6 - 14 & = & -20 & \text{or } -4 \\ 7 - 12 & = & -\phantom{2}5 & \text{or } -1 \end{array}$$

$$\therefore z = \frac{2 \times 63 - 4 \times 22 - 1 \times 18}{2 \times -1 - 4 \times -3 - 1 \times 5} = \frac{20}{5} = 4.$$

The value of $z$ being thus determined, the three equations may now, by substitution, be reduced to two, containing $x$ and $y$ only; or these unknowns may be found, independently of the value of $z$, by imitating the operation above.

# NOTE (B.) PAGE 154.

## DECOMPOSITION OF EQUATIONS OF THE FOURTH DEGREE.

THE method of decomposing certain equations of the fourth degree into quadratics, as taught in the text, may, with a little address, be rendered applicable to many forms which at first sight might appear to be beyond its reach.

There is a class of equations, called *recurring equations*, to which the process always applies with success. A recurring equation is one in which the coefficients recur, as in the following instances:

$$x^4 - 6x^3 + 2x^2 - 6x + 1 = 0$$
$$x^4 - 8x^3 + 3x^2 - 8x + 1 = 0$$

where we see that if the entire row of coefficients were to be written in reverse order, their respective places would be undisturbed.

We shall here decompose each of these into a pair of quadratics; and as the working process, so fully exhibited in the text, is nothing more than a new arrangement for the steps of a square root operation, we shall here, for the sake of variety, give to this operation the common form.

Extracting the square root of the first member of $x^4 - 6x^3 - 6x + 1$, we get the desired quadratic factors of the expression as follows:

$$\begin{array}{r l}
 & x^4 - 6x^3 + 2x^2 - 6x + 1\,(x^2 - 3x + 1 \\
 & x^4 \\
2x^2 - 3x) & \overline{\quad -6x^3 + 2x^2} \\
 & \quad -6x^3 + 9x^2 \\
2x^2 - 6x + 1) & \overline{\qquad\quad -7x^2 - 6x + 1} \\
 & \qquad\qquad 2x^2 - 6x + 1 \\
 & \qquad\quad \overline{-9x^2}
\end{array}$$

From this result we infer, that the proposed expression of the fourth degree is equivalent to

$$(x^2 - 3x + 1)^2 - 9x^2;$$

and, consequently, that it is formed by the multiplication of the two quadratic factors,

$$(x^2 - 3x + 1) + 3x,$$
$$\text{and } (x^2 - 3x + 1) - 3x;$$

so that the given equation may be replaced by the two quadratic equations,

$$x^2+1=0;\ x^2-6x+1=0$$

which are of course very easily solved.

Treating the equation $x^4-8x^3+3x^2-8x+1=0$, in the same way, we have,

$$\begin{array}{r l}
 & x^4-8x^3+\ 3x^2-8x+1\,(x^2-4x+1 \\
 & x^4 \\
2x^2-4x) & \quad -8x^3+\ 3x^2 \\
 & \quad -8x^3+16x^2 \\
 & \qquad\qquad -13x^2-8x+1 \\
 & \qquad\qquad\quad\ 2x^2-8x+1 \\
 & \qquad\qquad -15x^2
\end{array}$$

consequently, the proposed expression of the fourth degree is the same as

$$(x^2-4x+1+\sqrt{15}.x)(x^2-4x+1-\sqrt{15}.x)$$

so that the two component quadratics are,

$$x^2+(\sqrt{15}-4)x+1=0,$$
$$\text{and }\ x^2-(\sqrt{15}+4)x+1=0.$$

Let us now take a recurring equation of the fourth degree, in the most general form—namely,

$$x^4+ax^3+bx^2+ax+1=0.$$

Extracting the square root of the first member, as in the former particular instances, we have,

$$\begin{array}{r l}
 & x^4+ax^3+bx^2+ax+1\,(x^2+\tfrac{1}{2}ax+1 \\
 & x^4 \\
2x^2+\tfrac{1}{2}ax) & \quad ax^3+bx^2 \\
 & \quad ax^3+\tfrac{1}{4}a^2x^2 \\
2x^2+ax+1) & \quad (b-\tfrac{1}{4}a^2)x^2+ax+1 \\
 & \qquad\qquad\quad 2x^2+ax+1 \\
 & \quad (b-\tfrac{1}{4}a^2-2)x^2;
\end{array}$$

consequently, the first member of the equation is the same as

$$(x^2+\tfrac{1}{2}ax+1)^2-(\tfrac{1}{4}a^2-b+2)x^2$$

and, therefore, the component quadratics are,

$$x^2+(\tfrac{1}{2}a+\quad \tfrac{1}{4}a^2-b+2)x+1=0$$
$$x^2+(\tfrac{1}{2}a-\sqrt{\tfrac{1}{4}a^2-b+2})x+1=0.$$

It is plain, from the final step in the square root operation, that the method applies, whatever be the *signs* of $a$ and $b$. It is further obvious that, the final term in each quadratic being 1, two of the four roots are the reciprocals of the other two.

We shall now show, by an example or two, how, by the exercise of a little ingenuity, we may often secure a *trinomial* square for the final remainder of the square root process, when the method is applied to any equation of the fourth degree. The mode of proceeding, so as to get a *monomial* square for this remainder, is sufficiently illustrated in the text. When the suitable monomial remainder does not exist, a trinomial remainder, answering the desired object, may often be obtained, as in the following examples:

1. $x^4+2x^3+2x^2-6x-15=0$

$$
\begin{array}{rl}
 & x^4+2x^3+2x^2-6x-15\ (x^2+x+1 \\
 & \underline{x^4\qquad\qquad\qquad\qquad} \\
2x^2+x) & \qquad 2x^3+2x^2 \\
 & \qquad \underline{2x^3+\ \ x^2\qquad\qquad\quad} \\
2x^2+2x+1) & \qquad\qquad\quad x^2-6x-15 \\
 & \qquad\qquad\quad \underline{2x^2+2x+\ \ 1} \\
 & \qquad\qquad\quad \underline{-x^2-8x-16}
\end{array}
$$

This remainder is $-(x^2+8x+16)$, the expression between the brackets being a complete square.

It is always the object to secure for the remainder such a trinomial form, whenever we fail in obtaining for it a single term which is either a number, or else something-times $x^2$.

As the first member of the proposed equation is thus found to be equivalent to $(x^2+x+1)^2-(x+4)^2$,

it follows that the equation may be replaced by the form

$$\{(x^2+x+1)+(x+4)\}\{(x^2+x+1)-(x+4)\}=0;$$

and, therefore, by the two quadratics,

$$x^2+2x+5=0,\ x^2-3=0.$$

2. $x^4-x^3-8x^2+16x-8=0$

$$
\begin{array}{rl}
 & x^4-x^3-\ 8x^2+16x-8\ (x^2-\tfrac{1}{2}x-1 \\
 & \underline{x^4\qquad\qquad\qquad\qquad} \\
2x^2-\tfrac{1}{2}x) & -x^3-\ 8x^2 \\
 & \underline{-x^3+\ \tfrac{1}{4}x^2\qquad\qquad\qquad} \\
2x^2-x-1) & \qquad -\tfrac{33}{4}x^2+16x-8 \\
 & \qquad \underline{-\ \ 2x^2+\quad\ x+1} \\
 & \qquad \underline{-\tfrac{25}{4}x^2+15x-9}=-(\tfrac{5}{2}x-3)^2;
\end{array}
$$

consequently, the equation is the same as

$$(x^2-\tfrac{1}{2}x-1)^2-(\tfrac{5}{2}x-3)^2=0;$$

and, therefore, the component quadratics are

$$x^2+2x-4=0,\ x^2-3x+2=0.$$

The student must perceive, from these examples, that, in applying the present method of analysing an equation of the fourth degree into a pair of quadratic factors, all that is necessary is a little careful consideration in the management of the final step, in the operation of extracting the square root. He is to observe, first, whether there be any *number* which, put in the root's place for the final term, can cause either the first two or the last two terms of the dividend reached to become cancelled from the final remainder. A very little examination is sufficient to settle this point, as he must have already felt in working the examples in the present Key. If no such number exist, then he is to seek a number for the last term of the root, such that the final remainder resulting from it may be a complete trinomial square, as in the foregoing examples. The suitable number will, in general, be readily discovered, provided the equation really admit of the desired decomposition, without the aid of principles beyond the scope of quadratic equations. The suitable number *always exists*, and may always be found by aid of the solution of a *cubic* equation; but our purpose here is to effect the object, whenever possible, without resorting to this aid; and no method has hitherto been proposed which is successful, in so great a variety of different forms of equations, as the method so fully illustrated in the Algebra and in this Key.

3. $x^4+2x^3-16x^2+19x-6=0$

$$\begin{array}{rl}
 & x^4+2x^3-16x^2-19x-6\,(x^2+x-\tfrac{1}{2} \\
 & x^4 \\ \hline
2x^2+x) & 2x^3-16x^2 \\
 & 2x^3+\phantom{16}x^2 \\ \hline
2x^2+2x-\tfrac{1}{2}) & -17x^2+19x-6 \\
 & -\phantom{17}x^2-\phantom{19}x+\tfrac{1}{4} \\ \hline
 & -16x^2+20x-\tfrac{25}{4}=-(4x-\tfrac{5}{2})^2 \\ \hline
\end{array}$$

consequently, the first member of the equation is equivalent to

$$(x^2+x-\tfrac{1}{2})^2-(4x-\tfrac{5}{2})^2;$$

and, therefore, the equation itself is decomposable into the two quadratics,

$$x^2+5x-3=0,\ x^2-3x+2=0.$$

4. $x^4 + 2x^3 + 3x^2 + x + 3 = 0$

$$\begin{array}{rl}
 & x^4 + 2x^3 + 3x^2 + x + 3 \,(x^2 + x \\
 & x^4 \\
\hline
2x^2 + x) & \quad 2x^3 + 3x^2 \\
 & \quad 2x^3 + \phantom{3}x^2 \\
\hline
 & \qquad\qquad 2x^2 + x + 3.
\end{array}$$

Here, without proceeding further, we see that if the quadratic remainder now obtained were equated to zero, the roots would be imaginary; for four times the product of the extreme terms exceeds the square of the middle term (see Algebra, page 180). We therefore conclude at once that all the roots of the proposed equation are imaginary, the first member of it being equivalent to

$$(x^2 + x)^2 + (2x^2 + x + 3),$$

which is the sum of two quantities *necessarily positive*, whatever real value be put for $x$: it is impossible, therefore, that any *real* value of $x$ can ever render the expression equal to zero.

We shall terminate these remarks by showing how a general *criterion* may be obtained, such that when the condition of it is fulfilled, we may at once conclude that the equation tested by it must have all its roots imaginary; but instead of extracting the square root, as in the examples given above, we shall proceed by the more compact and convenient process employed in the Algebra.

Let the general equation of the fourth degree be

$$x^4 + ax^3 + bx^2 + cx + d = 0;$$

then, decomposing into factors, we have the pair,

$$x^2 \pm \sqrt{\{-ax^3 - bx^2 - cx - d\}}$$
$$x^2 + \tfrac{1}{2}ax \pm \sqrt{\{-(b - \tfrac{1}{4}a^2)x^2 - cx - d\}};$$

now, if the expression

$$(b - \tfrac{1}{4}a^2)x^2 + cx + d \ldots (A)$$

be always *positive*, whatever real value be put for $x$, then, as in the preceding particular example, the first member of the equation will become equivalent to the *sum* of two expressions, *both* of which are necessarily *positive* for every *real* value of $x$, and, consequently, can never become zero for any such value. But $(A)$ is always positive for real values of $x$, provided it fulfil the condition,

$$4\,(b - \tfrac{1}{4}a^2)\,d > c^2$$
$$\text{or } (4b - a^2)\,d > c^2;$$

$d$ being positive. This, therefore, is the criterion, the fulfilment of the condition of which implies the impossibility of all the roots of the equation. If the second term, $a x^3$, be absent from the first member, then the criterion takes the more simple form,

$$4bd > c^2;$$

and it is plain that if, instead of $>$, (the sign for "*greater than*,") the sign $=$ (of equality) be put, the same conclusion follows; for the first member of the equation will still be formed of the sum of two squares, as $(A)$ in that case will be a complete square.*

The student will, of course, keep in remembrance that although, when the above condition is fulfilled, the roots are all necessarily imaginary, yet it does not follow that they may not be imaginary though the criterion fail. In a quadratic equation, the criterion of imaginary roots holds directly and conversely. If the criterion (Algebra, page 180) be satisfied, the roots are imaginary; and if the roots are imaginary, the criterion must be satisfied: but the latter inference does not necessarily follow from what is done above.

## NOTE (C).

### NEW METHOD OF INVESTIGATING THE EXPONENTIAL THEOREM.

At page 239 of the Algebra, the development of $a^x$, according to the ascending powers of $x$, is shown to be

$$a^x = 1 + A x + \frac{A^2}{2} x^2 + \frac{A^3}{2.3} x^3 + \frac{A^4}{2.3.4} x^4 +, \text{\&c.}$$

where $A$ is a constant quantity depending entirely upon $a$. The actual determination of $A$ is, however, postponed to the article on Logarithms (p. 244); and the discovery of its value, by help of that theory, is not unattended by what a learner may consider some amount of difficulty. We therefore propose to give here a shorter and easier investigation of the Exponential Theorem.

By the Binomial Theorem,

---

* There is, however, one case of exception, namely, when the roots of $(A) = 0$ *also* satisfy the equation $x^2 + \frac{1}{2} a x = 0$, or $x + \frac{1}{2} a = 0$, in which case the proposed equation would have a pair of equal roots, each of the pair being $x = -\frac{1}{2} a$; but the other two roots would necessarily be imaginary.

$$(1+a)^m = 1 + ma + \frac{m(m-1)}{2}a^2 + \frac{m(m-1)(m-2)}{2.3}a^3 +, \text{\&c.}$$

$$\therefore \frac{(1+a)^m - 1}{m} = a + \frac{m-1}{2}a^2 + \frac{(m-1)(m-2)}{2.3}a^3 +, \text{\&c.}$$

This being an identical equation, any value whatever may of course be put for $m$. Put $m = 0$, and we have

$$\frac{(1+a)^0 - 1}{0} = a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4 +, \text{\&c.}$$

The zero may be replaced by the symbol $\frac{1}{\infty}$: introducing this change, therefore, and putting $a-1$ for $a$, since $a$ may be anything, we have,

$$\infty\,(a^{\frac{1}{\infty}} - 1) = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 +, \text{\&c.}$$

Call the right-hand member of this identity $A$, then

$$\infty\,(a^{\frac{1}{\infty}} - 1) = A\,;\ \therefore a^{\frac{1}{\infty}} = 1 + \frac{A}{\infty}\,;\ \therefore a = (1 + \frac{A}{\infty})^{\infty};$$

consequently, taking the power $x$ of each side of this equation, and assuming the truth of the Binomial Theorem in its general form, we have

$$a^x = 1 + \infty x \frac{A}{\infty} + \frac{\infty x(\infty x - 1)}{2}\left(\frac{A}{\infty}\right)^2 + \frac{\infty x(\infty x - 1)(\infty x - 2)}{2.3}\left(\frac{A}{\infty}\right)^3 +, \text{\&c.};$$

or, which is the same thing, observing that $\frac{1}{\infty} = 0$,

$$a^x = 1 + \frac{\infty}{\infty}Ax + \frac{\infty\,.\,\infty}{\infty^2}\,\frac{A^2x^2}{2} + \frac{\infty\,.\,\infty\,.\,\infty}{\infty^3}\,\frac{A^3x^3}{2.3} +, \text{\&c.}$$

$$= 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4} +, \text{\&c.}$$

where $A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 +$, &c.

The student will perceive that there is some novelty in the preceding reasoning: the symbol 0, or its equivalent $\frac{1}{\infty}$, plays an important part in the several steps, and is considered throughout as subject to the same rules of operation as all the other symbols of arithmetic. In dealing with expressions that are perfectly general, and therefore entirely unrestricted as to the numerical values of the algebraical characters which enter into them, it is plain that the whole range of the numbers of arithmetic, from 0 up to $\infty$, is available for the purpose of a suitable selection to accomplish the end in view. If an equation be true for *all* values of any arbitrary symbol entering it, it is necessarily true for the values 0 and $\infty$; and those who would maintain that these should be excluded from the operations of arithmetic, are bound to show at what small value, greater than zero, the available arithmetical scale of numbers commences, and at what great value, short of infinity, it terminates. But it would be futile to insist upon any such imaginary limitations. Zero-values are recognised as fully, and operated upon as freely, in the earlier parts of Algebra, as the other symbols of arithmetic: it is only in the more advanced departments of the subject that analysts entertain any scruples about the introduction of zero and infinity. If one side of a simple equation consist of significant quantities, and the other side be zero, no one ever feels any hesitation about applying the same operations to *both* sides; and if zero be admitted to the claims of Algebra at one stage of its progress, there can be no valid reason for ignoring its existence in any of the subsequent investigations of that science.

As the foregoing instance shows, a cautious use of this symbol, in rigid conformity to the laws of Algebra, which govern operations with symbols in general, may lead to interesting results; and the inquiring student, in the course of his further analytical progress, will meet with many important theorems that cannot be established, in a perfectly secure and unobjectionable manner, without bringing zero into full subjection to the general laws of analysis.

If, in the foregoing expression for $A$, $a$ be supposed to take a value such that $A$ may become equal to 1, then representing that value of $a$ by $e$, the Exponential Theorem becomes

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +, \&c.$$

And, by the aid of this, we may readily obtain the development of the expression

$$\frac{x}{e^x - 1}$$

a development of frequent use in some of the higher departments of analysis. But it is introduced here chiefly for the purpose of illustrating a method of applying the principle of *indeterminate coefficients*, which will often dispense with much of the symbolical work usually introduced into the process.

It is plain that if the preceding series for $e^x$ were actually substituted in the expression proposed, the leading term of the required development, arranged according to the ascending powers of $x$, would be 1. We may therefore assume that

$$\frac{x}{e^x - 1} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.;$$

and consequently that

$$x = \left(x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.\right) \times \\ (1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.)$$

Now, as this is an *identical* equation, the product furnished by the second member must be such, that the coefficients of the powers of $x$, after $x$ itself, must be severally zero. It is evident that the coefficient of $x^2$ may be got by fancying the first two terms, $1 + Ax$, of the multiplier to be *reversed;* that is, to be written $Ax + 1$, and then multiplying each by the term above it. In like manner, the coefficient of $x^3$ is obtained by reversing the *three* leading terms of the multiplier, and proceeding in a similar way; and so on, to any extent. We thus get

$$A + \tfrac{1}{2} = 0$$

$$B + \frac{A}{2} + \frac{1}{2.3} = 0$$

$$C + \frac{B}{2} + \frac{A}{2.3} + \frac{1}{2.3.4} = 0$$

$$D + \frac{C}{2} + \frac{B}{2.3} + \frac{A}{2.3.4} + \frac{1}{2.3.4.5} = 0,$$

&c. &c. &c.

These equations establish the law of the coefficients, and for their actual values give

$$A = -\tfrac{1}{2},\ B = \frac{1}{2.6},\ C = 0,\ D = -\frac{1}{2.3.4.30},\ \&c.;$$

so that the development of the proposed expression is

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{1}{6}\cdot\frac{x^2}{2} - \frac{1}{30}\cdot\frac{x^4}{2.3.4} + \&c.$$

It is usual to represent the numbers

$$-\frac{1}{2},\ \frac{1}{6},\ 0,\ -\frac{1}{30},\ \&c.$$

by the symbols $B_1,\ B_2,\ B_3,\ B_4,$ &c.

and to call them the *Numbers of Bernoulli*, on account of the use first made of them by the analyst of that name.

Employing this notation, the identical equation above referred to would be

$$x = (x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \&c.) \times$$

$$(1 + 2B_1\frac{x}{2} + 3B_2\frac{x^2}{2.3} + 4B_3\frac{x^3}{2.3.4} + 5B_4\frac{x^4}{2.3.4.5} + \&c.)$$

from which we deduce, by multiplying as already explained, the following conditions; namely,

$$\begin{aligned} 2B_1 + 1 &= 0 \\ 3B_2 + 3B_1 + 1 &= 0 \\ 4B_3 + 6B_2 + 4B_1 + 1 &= 0 \\ 5B_4 + 10B_3 + 10B_2 + 6B_1 + 1 &= 0 \\ 6B_5 + 15B_4 + 20B_3 + 15B_2 + 6B_1 + 1 &= 0. \end{aligned}$$

In these expressions, when taken in reverse order, we recognise the law of the coefficients in the development of a binomial, the final coefficient in each case being absent. If therefore, in the development of $(1+B)^n$, we only remember to write the several exponents of $B$, as they arise, *below*, instead of *above*, that symbol, we may express the general law, under which the preceding particular cases come, by the following form; namely,

$$(1 + B)^n - B_n = 0,$$

which, as $n$ is necessarily a whole number, may always be expeditiously developed by deducing the several coefficients, one after another, in the usual way, all of them being integral.

If we agree that the development of $(1+B)_n$ should differ from that of the binomial $(1+B)^n$ only by the position of the index $n$, then the general form would be

$$(1 + B)_n - B_n = 0.$$

Thus for $n = 4$, we should have

$$(1 + B)_4 = 1 + 4B_1 + 6B_2 + 4B_3 + B_4;$$

$$\therefore (1 + B)_4 - B_4 = 1 + 4B_1 + 6B_2 + 4B_3.$$

The method employed above, in the management of the indeterminate or conditional coefficients, may always be applied with advantage in the multiplication of any two series together, which proceed according to the powers of $x$, more especially when the coefficient of a specified power of $x$ in the product is alone required; taking care, of course, to replace absent terms in either series by zeros. Thus, the coefficient of $x^4$, in the product,

$$\begin{array}{l}(a + bx + cx^2 + dx^3 + ex^4 + \&c.) \times \\ (a_1 + b_1x + c_1x^2 + d_1x^3 + e_1x^4 + \&c.),\end{array}$$

is at once found, as above, to be

$$ae_1 + bd_1 + cc_1 + db_1 + ea_1.$$

In like manner, if there be three series, we may find, as before explained, all the terms in the product of two, up to that involving the specified power inclusive; and then, as here, introduce the third series to find the coefficient of that power in the product; and so on, for any number of series.

The method may obviously be varied by multiplying the coefficients in cross order.

We here terminate these supplementary investigations. They might have been suppressed altogether, without diminishing the claim of the present work to its title as a KEY to the "Introduction to Algebra;" but one object of the elementary treatise just mentioned, is to stimulate the learner to further inquiry,—to create in him a taste for analytical pursuits,—and to foster a desire for more advanced attainments. One of the most important ends of Algebra is the accomplishment of the general solution of equations; and although, as stated in the Preface to this book, we do not think it judicious, in an elementary writer on *any* subject, to be continually decoying the learner from the authorized beaten path into tracks of his own; yet, to promote the purposes hinted at above, and to suggest practical facilities in the management of things of acknowledged difficulty, of which he must necessarily treat, he may perhaps, without incurring censure, deliver what he considers to be improvements, in a few supplementary pages, as we have done here.

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